9.3 Perform Reflections

Obj.: Reflect a figure in any given line.

Key Vocabulary
• Line of reflection - A reflection is a transformation that uses a line like a mirror to reflect an image. The mirror line is called the line of reflection.
• Reflection - A reflection uses a line of reflection to create a mirror image of the original figure.

*****KEY CONCEPT*****
Coordinate Rules for Reflections
• If \((a, b)\) is reflected in the \(x\)-axis, its image is the point \((a, -b)\).
• If \((a, b)\) is reflected in the \(y\)-axis, its image is the point \((-a, b)\).
• If \((a, b)\) is reflected in the line \(y = x\), its image is the point \((b, a)\).
• If \((a, b)\) is reflected in the line \(y = -x\), its image is the point \((-b, -a)\).

Reflection Theorem
A reflection is an isometry.

EXAMPLE 1 Graph reflections in horizontal and vertical lines
The vertices of \(\triangle ABC\) are \(A(1, 2)\), \(B(3, 0)\), and \(C(5, 3)\). Graph the reflection of \(\triangle ABC\) described.

a. In the line \(n: x = 2\)
b. In the line \(m: y = 3\)

Solution

EXAMPLE 2 Graph a reflection in \(y = x\)
The endpoints of \(\overline{CD}\) are \(C(-2, 2)\) and \(D(1, 2)\). Reflect the segment in the line \(y = x\). Graph the segment and its image.

Solution
EXAMPLE 3 Graph a reflection in \( y = -x \)
Reflect \( \overline{CD} \) from Example 2 in the line \( y = -x \). Graph \( \overline{CD} \) and its image.

Solution

Use the coordinate rule for reflecting in the line \( y = -x \).

\[(a, b) \rightarrow (-b, -a)\]

\( C(-2, 2) \rightarrow C'(2, -2) \)

\( D(1, 2) \rightarrow D'(2, 1) \)

EXAMPLE 4 Find a minimum distance

Tools Workers are retrieving tools that they need for a project. One will enter the building at point A and the other at point B. Where should they park on driveway \( m \) to minimize the distance they will walk?

Solution

Reflect \( B \) in line \( m \) to obtain \( B' \).
Then draw \( \overline{AB'} \). Label the intersection of \( \overline{AB'} \) and \( m \) as \( C \). Because \( \overline{AB'} \) is the shortest distance between \( A \) and \( B' \) and \( BC = \overline{B'C} \), park at point \( C \) to minimize the combined distance, \( AC + BC \), they have to walk.
9.3 Cont.

**Checkpoint** Complete the following exercise.

1. Graph the reflection of \( \triangle ABC \) from Example 1 in the line \( y = 2 \).

![Graph of \( y = 2 \)](image)

**Checkpoint** The endpoints of \( \overline{JK} \) are \( J(-1, -2) \) and \( K(1, -2) \). Reflect the segment in the given line. Graph the segment and its image.

2. \( y = x \)

![Graph of \( y = x \)](image)

3. \( y = -x \)

![Graph of \( y = -x \)](image)

**Checkpoint** Complete the following exercise.

4. In Example 4, reflect \( A \) in line \( m \). What do you notice?

![Graph of \( m \) and \( A \)](image)

You obtain the same point \( C \) at which to park.
9.4 Perform Rotations

Obj.: Rotate figures about a point.

Key Vocabulary
- Center of rotation - A rotation is a transformation in which a figure is turned about a fixed point called the center of rotation.
- Angle of rotation - Rays drawn from the center of rotation to a point and its image form the angle of rotation.
- Rotation - A rotation turns a figure about a fixed point, called the center of rotation.

A rotation about a point \( P \) through an angle of \( x^\circ \) maps every point \( Q \) in the plane to a point \( Q' \) so that one of the following properties is true:
- If \( Q \) is not the center of rotation \( P \), then \( QP = Q'P \) and \( m \angle QPQ' = x^\circ \), or
- If \( Q \) is the center of rotation \( P \), then the image of \( Q \) is \( Q' \).

Rotations can be clockwise or counterclockwise. In this chapter, all rotations are counterclockwise.

Coordinate Rules for Rotations about the Origin
When a point \((a, b)\) is rotated counterclockwise about the origin, the following are true:
1. For a rotation of \(90^\circ\), \((a, b) \rightarrow (-b, a)\).
2. For a rotation of \(180^\circ\), \((a, b) \rightarrow (-a, -b)\).
3. For a rotation of \(270^\circ\), \((a, b) \rightarrow (b, -a)\).

Rotation Theorem
A rotation is an isometry.

\[ \Delta ABC \cong \Delta A'B'C' \]
EXAMPLE 1 Draw a rotation
Draw a 150° rotation of \( \triangle ABC \) about \( P \).
Solution

\[ \begin{array}{c}
\text{Step 1} \quad \text{Draw a segment from } A \text{ to } P. \\
\text{Step 2} \quad \text{Draw a ray to form a } 150° \\
\text{angle with } \overrightarrow{PA}. \\
\text{Step 3} \quad \text{Draw } A' \text{ so that } PA' = PA. \\
\text{Step 4} \quad \text{Repeat Steps 1–3 for each vertex. Draw } \triangle A'B'C'.
\end{array} \]

EXAMPLE 2 Rotate a figure using the coordinate rules
Graph quadrilateral \( KLMN \) with vertices \( K(3, 2) \), \( L(4, 2) \), \( M(4, -3) \), and \( N(2, -1) \). Then rotate the quadrilateral 270° about the origin.
Solution
Graph \( KLMN \). Use the coordinate rule for a 270° rotation to find the images of the vertices.

\[
\begin{align*}
(a, b) & \rightarrow (b, -a) \\
K(3, 2) & \rightarrow K'(2, -3) \\
L(4, 2) & \rightarrow L'(2, -4) \\
M(4, -3) & \rightarrow M'(-3, -4) \\
N(2, -1) & \rightarrow N'(-1, -2)
\end{align*}
\]
Graph the image \( K'L'M'N' \).

EXAMPLE 3 Find side lengths in a rotation
The quadrilateral is rotated about \( P \). Find the value of \( y \).
Solution
By Theorem 9.3, the rotation is an isometry, so corresponding side lengths are equal. Then \( 3x = 6 \), so \( x = 2 \). Now set up an equation to solve for \( y \).

\[
\begin{align*}
7y & = 3x + 1 \\
7y & = 3(2) + 1 \\
y & = 1
\end{align*}
\]

Corresponding lengths in an isometry are equal.
Substitute 2 for \( x \).
Solve for \( y \).
9.4 Cont.

**Checkpoint** Complete the following exercise.

1. Draw a $60^\circ$ rotation of $\triangle GHJ$ about $P$.

2. Graph $KLMN$ in Example 2. Then rotate the quadrilateral $90^\circ$ about the origin.

4. The triangle is rotated about $P$. Find the value of $b$.
   
   $b = 4$
9.5 Apply Compositions of Transformations

Obj.: Perform combinations of two or more transformations.

Key Vocabulary
• Glide reflection - A translation followed by a reflection can be performed one after the other to produce a glide reflection.
• Composition of transformations - When two or more transformations are combined to form a single transformation, the result is a composition of transformations.

A glide reflection is a transformation in which every point \( P \) is mapped to a point \( P' \) by the following steps.

**STEP 1** First, a translation maps \( P \) to \( P' \).
**STEP 2** Then, a reflection in a line \( k \) parallel to the direction of the translation maps \( P' \) to \( P'' \).

Composition Theorem
The composition of two (or more) isometries is an isometry.

**Reflections in Parallel Lines Theorem**
If lines \( k \) and \( m \) are parallel, then a reflection in line \( k \) followed by a reflection in line \( m \) is the same as a translation.
If \( P' \) is the image of \( P \), then:
1. \( PP' \) is perpendicular to \( k \) and \( m \), and
2. \( PP' = 2d \), where \( d \) is the distance between \( k \) and \( m \).

**Reflections in Intersecting Lines Theorem**
If lines \( k \) and \( m \) intersect at point \( P \), then a reflection in \( k \) followed by a reflection in \( m \) is the same as a rotation about point \( P \). The angle of rotation is \( 2x^\circ \), where \( x^\circ \) is the measure of the acute or right angle formed by \( k \) and \( m \).

\[ \angle BPB'' = 2x^\circ \]

**EXAMPLE 1** Find the image of a glide reflection
The vertices of \( \triangle ABC \) are \( A(2, 1) \), \( B(5, 3) \), and \( C(6, 2) \). Find the image of \( \triangle ABC \) after the glide reflection.
Translation: \((x, y) \rightarrow (x - 8, y)\)
Reflection: in the \( x \)-axis

Solution
Begin by graphing \( \triangle ABC \). Then graph \( \triangle A'B'C' \) after a translation 8 units left. Finally, graph \( \triangle A''B''C'' \) after a reflection in the \( x \)-axis.
EXAMPLE 2 Find the image of a composition
The endpoints of \( \overline{CD} \) are \( C(-1, 6) \) and \( D(-1, 3) \). Graph the image of \( \overline{CD} \) after the composition.
Reflection: in the y-axis
Rotation: 90° about the origin

Solution

Unless you are told otherwise, do the transformations in the order given.

Step 1 Graph \( \overline{CD} \).
Step 2 Reflect \( \overline{CD} \) in the y-axis. \( \overline{C'D'} \) has endpoints \( C'(2, 6) \) and \( D'(1, 3) \).
Step 3 Rotate \( \overline{C'D'} \) 90° about the origin. \( \overline{C''D''} \) has endpoints \( C''(-6, 2) \) and \( D''(-3, 1) \).

EXAMPLE 3 Use Reflect // lines Thm
In the diagram, a reflection in line \( k \) maps \( \overline{GF} \) to \( \overline{G'F'} \). A reflection in line \( m \) maps \( \overline{G'F'} \) to \( \overline{G''F''} \). Also, \( FA = 6 \) and \( DF'' = 3 \).

a. Name any segments congruent to each segment: \( \overline{GF} \), \( \overline{FA} \), and \( \overline{GB} \).
b. Does \( AD = BC \)? Explain.
c. What is the length of \( \overline{GG''} \)?

Solution

a. \( GF \cong G'F' \) and \( GF \cong G'F' \). \( FA \cong FA \).

b. \( GB \cong G'B \).

By the properties of reflections, \( FA = 6 \) and \( F'D = 3 \). Theorem 9.5 implies that \( GG'' = FF'' = 2 \cdot AD \), so the length of \( GG'' \) is \( 2(6 + 3) \), or 18 units.

EXAMPLE 4 Use Reflect int. lines Thm
In the diagram, the figure is reflected in line \( k \). The image is then reflected in line \( m \). Describe a single transformation that maps \( F \) to \( F' \).

Solution

The measure of the acute angle formed between lines \( k \) and \( m \) is 80°. So, by Theorem 9.6, a single transformation that maps \( F \) to \( F'' \) is a 160° rotation about point \( P \).

You can check that this is correct by tracing lines \( k \) and \( m \) and point \( F \), then rotating the point 160°.
9.5 Cont.

**Checkpoint** Complete the following exercises.

1. Suppose \( \triangle ABC \) in Example 1 is translated 5 units down, then reflected in the y-axis. What are the coordinates of the vertices of the image?

   \[ A'(-2, -4), \quad B'(-5, -2), \quad C'(-6, -3) \]

2. Graph \( \overline{CD} \) from Example 2. Do the rotation first, followed by the reflection.

3. In Example 3, suppose you are given that \( BC = 10 \) and \( G'F' = 6 \). What is the perimeter of quadrilateral \( GG''F''F \)?

   52 units

4. In the diagram below, the preimage is reflected in line \( k \), then in line \( m \). Describe a single transformation that maps \( G \) to \( G'' \).

   \[ 136^\circ \text{ rotation about point } P \]
9.6 Identify Symmetry

Obj.: Identify line and rotational symmetries of a figure.

Key Vocabulary
- Line symmetry - A figure in the plane has line symmetry if the figure can be mapped onto itself by a reflection in a line.
- Line of symmetry - This line of reflection is a line of symmetry, such as line $m$ at the right. A figure can have more than one line of symmetry.
- Rotational symmetry - A figure in a plane has rotational symmetry if the figure can be mapped onto itself by a rotation of $180^\circ$ or less about the center of the figure.
- Center of Symmetry - This point (the center of the figure) is the center of symmetry. Note that the rotation can be either clockwise or counterclockwise.

The figure above also has point symmetry, which is $180^\circ$ rotational symmetry.

EXAMPLE 1 Identify lines of symmetry

How many lines of symmetry does the figure have?

Solution

a.  

b.  

c.  

a. Two lines of symmetry  
b. Five lines of symmetry  
c. One line of symmetry

Notice that the lines of symmetry are also lines of reflection.
EXAMPLE 2 Identify rotational symmetry

Does the figure have rotational symmetry? If so, describe any rotations that map the figure onto itself.

a. Square    b. Regular hexagon    c. Kite

Solution

a. The square has rotational symmetry. The center is the intersection of the diagonals. Rotations of \(90^\circ\) or \(180^\circ\) about the center map the square onto itself.

b. The regular hexagon has rotational symmetry. The center is the intersection of the diagonals. Rotations of \(60^\circ\), \(120^\circ\), or \(180^\circ\) about the center all map the hexagon onto itself.

c. The kite does not have rotational symmetry because no rotation of \(180^\circ\) or less maps the kite onto itself.

EXAMPLE 3 Identify symmetry

Identify the line symmetry and rotational symmetry of the equilateral figure at the right.

Solution

The figure has line symmetry. Two lines of symmetry can be drawn for the figure.

For a figure with \(s\) lines of symmetry, the smallest rotation that maps the figure onto itself has the measure \(\frac{360^\circ}{s}\). So, the figure has \(\frac{360^\circ}{2}\), or \(180^\circ\) rotational symmetry.
9.6 Cont.

**Checkpoint** How many lines of symmetry does the figure have?

1. [Figure 1] 1
2. [Figure 2] 4

In Exercises 3 and 4, does the figure have rotational symmetry? If so, describe any rotations that map the figure onto itself.

3. [Figure 3] yes; 90° or 180° about the center
4. [Figure 4] no

5. Describe the lines of symmetry and rotational symmetry of the figure at the right.

8 lines of symmetry, 4 through the convex vertices and 4 through the concave vertices; 45°, 90°, 135°, or 180° about the center
9.7 Identify and Perform Dilations

**Obj.:** Use drawing tools to draw dilations.

**Key Vocabulary**
- **Dilation** - A dilation is a transformation that stretches or shrinks a figure to create a similar figure.
- **Reduction** - If $0 < k < 1$, the dilation is a reduction.
- **Enlargement** - If $k > 1$, the dilation is an enlargement.

**EXAMPLE 1 Identify dilations**
Find the scale factor of the dilation. Then tell whether the dilation is a reduction or an enlargement.

**Solution**

**a.**
Because $\frac{CP'}{CP} = \frac{10}{6}$, the scale factor is $k = \frac{5}{3}$. The image $P'$ is an enlargement.

**b.**
Because $\frac{CP'}{CP} = \frac{11}{22}$, the scale factor is $k = \frac{1}{2}$. The image $P'$ is a reduction.

**EXAMPLE 2 Draw a dilation**
Draw and label $\square LMNP$. Then construct a dilation of $\square LMNP$ with point $L$ as the center of dilation and a scale factor of $\frac{1}{2}$.

**Solution**

**Step 1**
Draw $LMNP$. Draw rays from $L$ through vertices $M, N,$ and $P$.

**Step 2**
Open the compass to the length of $LM$. Locate $M'$ on $LM$ so $LM' = \frac{1}{2} (LM)$. Locate $N'$ and $P'$ the same way.

**Step 3**
Add a second label $L'$ to point $L$. Draw the sides of $L'M'N'P'$. 

![Diagram of the construction of a dilation]
EXAMPLE 4 Use scale factor multiplication in a dilation
The vertices of quadrilateral ABCD are A(−3, 0), B(0, 6), C(3, 6), and D(3, 3). Find the image of ABCD after a dilation with its center at the origin and a scale factor of \( \frac{1}{3} \). Graph ABCD and its image.
Solution

EXAMPLE 5 Find the image of a composition
The vertices of \( ▲KLM \) are \( K(−3, 0) \), \( L(−2, 1) \), and \( M(−1, −1) \). Find the image of \( ▲KLM \) after the given composition.
Translation: \((x, y) \rightarrow (x + 4, y + 2)\)
Dilation: centered at the origin with a scale factor of 2
Solution

Step 1 Graph the preimage \( ▲KLM \) in the coordinate plane.
Step 2 Translate \( ▲KLM \) 4 units to the right and 2 units up. Label it \( ▲K'L'M' \).
Step 3 Dilate \( ▲K'L'M' \) using the origin as the center and a scale factor of 2 to find \( ▲K''L''M'' \).
Checkpoint Complete the following exercise.

1. In a dilation, $CP' = 4$ and $CP = 20$. Tell whether the dilation is a reduction or an enlargement and find its scale factor.
   
   reduction; $\frac{1}{5}$

2. Draw and label $\triangle PQR$. Then construct a dilation of $\triangle PQR$ with $P$ as the center of dilation and a scale factor of 2.
   
   Sample answer:

5. The vertices of $\triangle RST$ are $R(-4, 3), S(-1, -2)$, and $T(2, 1)$. Use scalar multiplication to find the vertices of $\triangle R'S'T'$ after a dilation with its center at the origin and a scale factor of 2.

   $'(8, 6), S'(2, 4), T'(4, 2)$

6. A segment has the endpoints $C(-2, 2)$ and $D(2, 2)$. Find the image of $CD$ after a $90^\circ$ rotation about the origin followed by a dilation with its center at the origin and a scale factor of 2.

   $C''(4, 4), D''(4, 4)$