Topic 3: Logic

IB Math Studies
Mr. Steponic
Review Sheet for Chapters Nine and Ten: Sets and Logic

You should be able to do the following things on the test:

Translate between verbal propositions, symbolic language, and Venn diagrams
Use compound logic statements: implication $\Rightarrow$ and equivalence $\Leftrightarrow$
Use compound logic statements: negation $\neg$ and conjunction $\wedge$
Use compound logic statements: disjunction $\vee$ and exclusive disjunction $\underline{\vee}$
Use truth tables to provide proofs for complex logical statements
Define inverse, converse, and contrapositive
Use compound logic statements: inverse, converse, and contrapositive
Define logical equivalence, tautology, and contradiction
Use truth tables to provide proofs for logical equivalence, tautology, and contradiction

---

1. Use the following truth table to answer the questions.

1	2	3	4	5	6
p	q	$p \wedge q$	$\neg p$	$p \vee q$	$(p \vee q) \wedge (\neg p \wedge \neg q)$
T	T	T	F	T	F
T	F	F	F		F
F	T	F	T	T	F
F	F	T	F		F

a) The truth table contains two entries which are incorrect. One is in column three and the other is in column four. Circle the two incorrect entries.

b) Fill in the two missing values in column five.

c) How would you describe the statement represented by the values in column six?

2. Complete the truth table below and explain its significance.

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$\neg r$	$(p \Rightarrow q) \wedge (q \Rightarrow r) \wedge \neg r$	$\neg p$	$[(p \Rightarrow q) \wedge (q \Rightarrow r) \wedge \neg r] \Rightarrow \neg p$
T	T	T						T
T	T	F						T
T	F	T						T
T	F	F						T
F	T	T						T
F	T	F						T
F	F	T						T
F	F	F						T

3. Consider each of the following statements:

p: Alex is from Uruguay
q: Alex is a scientist
r: Alex plays the flute

a) Write each of the following arguments in symbols:

i) If Alex is not a scientist, then he is not from Uruguay.

ii) If Alex is a scientist, then he is either from Uruguay or plays the flute.

b) Write the following argument in words.

$\sim r \Rightarrow \sim(q \vee p)$

4.

Let $\mathscr{E} = \{x : 1 \leqslant x < 17 , x \in \mathbb{N}\}$.

P , Q and R are the subsets of $\mathscr{E}$ such that

$$P = \{\text{multiples of four}\} ;$$
$$Q = \{\text{factors of } 36\} ;$$
$$R = \{\text{square numbers}\} .$$

(a) List the elements of

(i) $\mathscr{E}$;

(ii) $P \cap Q \cap R$. *[2 marks]*

(b) Describe in words the set $P \cup Q$. *[1 mark]*

(c) (i) Draw a Venn diagram to show the relationship between sets P, Q and R . *[2 marks]*

(ii) Write the elements of $\mathscr{E}$ in the appropriate places on the Venn diagram. *[3 marks]*

(d) Let p , q and r be the statements

p : x is a multiple of four;
q : x is a factor of 36 ;
r : x is a square number.

(i) Write a sentence, in words, for the statement;

$$(p \vee r) \wedge \neg q$$ *[2 marks]*

(ii) Shade the region on your Venn diagram in part (c)(i) that represents $(p \vee r) \wedge \neg q$. *[1 mark]*

(iii) (a) Use a truth table to determine the values of $(p \vee r) \wedge \neg q$. Write the first three columns of your truth table in the following format.

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

[3 marks]

(b) Write down one possible value of x for which $(p \vee r) \wedge \neg q$ is true. *[1 mark]*

IB Math Studies
Introduction to Symbolic Logic: Negation, Conjunction, Disjunction

There is a type of reasoning that produces results based on certain laws of logic. For example, consider the following argument:

1. If you read the *Times*, then you are well-informed.
2. You read the *Times*.
3. Therefore, you are well-informed.

If you accept statements 1 and 2 as true, then you *must also* accept statement 3 as true. Statements 1 and 2 are called propositions. Statement 3 is called the conclusion.

The truth value of a simple proposition is either true (T) or false (F), but not both.

There are many ways in which simple propositions can be combined to form compound statements. Different combinations are formed by using words called connectives to join the statements. The most common connectives are and, or and not.

Propositions: p: John is 2 meters tall
q: John plays basketball

Connector	English Sentence	Symbol	Symbolic Sentence	Venn Diagram
not				
not				
and conjunction				
or disjunction				

Example 1: Write the following using symbolic logic.
Clearly define each set of propositions.

a) Tom is taking math and Mary is taking a physics course.
b) Ann is passing math but she is failing English.
c) We stop inflation or we increase wages.
d) Today is not Friday.

Example 2: Consider the following statements

p: Good mathematics students go to good universities.
q: Good music students are good mathematics students.
r: Students who go to good universities get good jobs.

a) From these statements, write two valid conclusions.
b) Write in words each of the following

i) $\sim q$
ii) $p \wedge r$

Example 3: Consider the two statements

p: Sherlock Holmes is alive.
q: Sherlock Holmes lives in London.

Write the following statements in symbolic form

a) Sherlock Holmes is alive and he lives in London.
b) Either Sherlock Holmes lives in London or he is alive.
c) Sherlock Holmes is neither alive nor does he live in London.
d) It is not the case that Sherlock Holmes is alive and he lives in London.

As you can see, these propositions and compound statements can get very complicated. A truth table can help us look at all the possible outcomes of compound statements.

Conjunction: $p \wedge q$ means "both p and q"

Suppose you are offered a position in a firm that requires both

p: An applicant must be at least 18 years of age AND

q: An applicant must be a college graduate

To get the job, you must meet both requirements.

p	q	$p \wedge q$

A conjunction is true when ____.
Otherwise, it is false.

Disjunction: $p \vee q$ means "either p or q or both"

Suppose you are offered a position in a firm that requires either

p: An applicant must be at least 18 years of age OR

q: An applicant must be a college graduate

To get the job, you must meet at least one requirement.

p	q	$p \vee q$

A disjunction is true when ____.
Otherwise, it is false.

Negation: ~p means "not p"

In English, we usually negate a statement by using the word not. If the original statement is true, the negation is false. If the original statement is false, then the negation is true.

p	$\sim p$

Try it:

Two propositions p and q are defined as follows.

p: Jones passed this course.
q: Smith passed this course.

a) Write in symbolic form
   i) Neither Jones nor Smith passed the course

   ii) It is not the case that Jones and Smith both passed the course

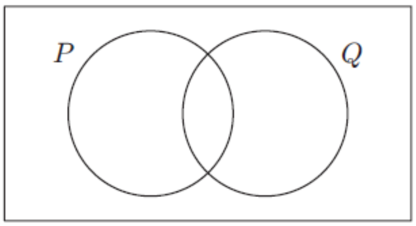
b) Complete the following truth table for the logic statement $\sim p \vee q$.

p	q	$\neg p$	$\neg p \vee q$
T	T		
T	F		
F	T		
F	F		

IB Math Studies
Implication, Exclusive Disjunction, and Truth Tables

Last class we looked at the "disjunction / or" compound statement.

Disjunction: $p \vee q$ means "either p or q or both"



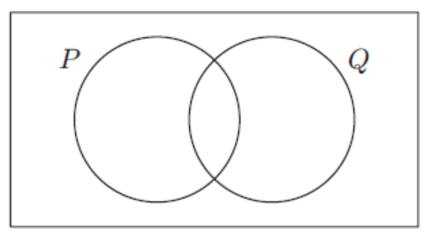
p	q	$p \vee q$

A disjunction is true when ____.
Otherwise, it is false.

An extension of this idea is the <u>exclusive disjunction</u>.

Exclusive Disjunction: $p \veebar q$ means "p or q, but not both"

p	q	$p \veebar q$



An exclusive disjunction is true when __________________________________.
Otherwise, it is false.

p	q	$p \veebar q$	$q \wedge (p \veebar q)$	$(p \veebar q) \vee p$
T	T			
T	F			
F	T			
F	F			

Implication: $p \Rightarrow q$ means "If p, then q."

If two propositions can be linked with "If…, then…", then we have an implication. The implication "if p, then q" is written as $p \Rightarrow q$.

Translate each of these into written if-then form.

a) All healthy dogs have four legs.

b) All triangles are polygons.

c) Brushing with Smiles Toothpaste gives you fewer cavities.

d) To qualify for the loan, you must have an income of at least $40,000.

Try this:

Let the propositions p, q, and r be defined as:

p: Matthew arrives home before six o'clock
q: Matthew cooks dinner
r: Jill washes the dishes

a) Express the following statement in logical form:
If Matthew arrives home before six o'clock then he will cook dinner.

b) Write the following logic statement in words:
$\sim q \Rightarrow \sim r$

Truth Tables

p: I do well on my test.
q: I get a sticker.

$p \Rightarrow q$ would be written as: ____.

Let's construct the truth table for implication:

I do well on my test. I get a sticker.
I do well on my test. I don't get a sticker.
I don't do well on my test. I get a sticker.
I don't do well on my test. I don't get a sticker.

p	q	$p \Rightarrow q$

You might think about an implication like making a promise:

If the promise is broken, the implication is false.
Otherwise, the implication is true.

IB Practice A

Complete the truth table for the compound proposition $(p \wedge \sim q) \Rightarrow (p \vee q)$.

p	q	$\neg q$	$(p \wedge \neg q)$	$(p \vee q)$	$(p \wedge \neg q) \Rightarrow (p \vee q)$
T	T	F	F		
T	F	T	T		
F	T	F		T	
F	F		F	F	

IB Practice B

Let p stand for the proposition "I will walk to school". Let q stand for the proposition "the sun is shining".

a) Write the following statements in symbolic logic form:

i) If the sun is shining, then I will walk to school.
ii) If I do not walk to school, then the sun is not shining.

b) Complete the table below by filling in the three empty columns.

p	q	$p \wedge q$	$p \vee q$	$\neg p$	$(p \vee q) \wedge \neg p$	$(p \vee q) \wedge \neg p \Rightarrow q$
T	T	T	T			
T	F	F	T			
F	T	F	T			
F	F	F	F			

IB Practice C

Consider the following statements:

p: students work hard
q: students will succeed

a) Write the following proposition in symbols using p, q, and logical connectives.

If students do not work hard, then they will not succeed.

b) Complete the following truth table, related to the proposition in part (a).

p	q			
T	T			
T	F			
F	T			
F	F			

IB Math Studies
Inverse, converse, contrapositive

Recall the implication example from last class:

p: a dog is healthy
q: a dog has four legs

$$p \Rightarrow q$$

If a dog is healthy, then it has four legs.

What other if-then statements could we write using these propositions?

A.

Is this statement necessarily true?

B.

Is this statement necessarily true?

C.

Is this statement necessarily true?

The statements we just looked at show us three other statements associated with an initial implication:

Initial implication: $p \Rightarrow q$

The <u>inverse</u> <u>*negates*</u> the initial implication: $\sim p \Rightarrow \sim q$ [turn upside down]

The <u>converse</u> <u>*reverses the order*</u> of the initial implication: $q \Rightarrow p$ [turn around]

The <u>contrapositive</u> <u>*reverses the order and negates*</u> the initial implication: $\sim q \Rightarrow \sim p$

---

a) Write the implication, the inverse, the converse, and the contrapositive. Use both words and symbolic logic.

p: It is raining
q: I will bring an umbrella

Implication:

Inverse:

Converse:

Contrapositive:

b) Write the inverse, converse, and contrapositive of $r \Rightarrow \sim s$.

Inverse:

Converse:

Contrapositive:

Just because an implication is true, it does not mean we can assume its inverse, converse, or contrapositive are also true.

Implication: If a dog is healthy, then it has four legs

Inverse: If a dog is not healthy, then it does not have four legs.

Converse: If a dog has four legs, then it is healthy.

Contrapositive: If a dog does not have four legs, then it is not healthy.

Implication: If it is raining, then I will bring an umbrella.

Inverse: If it is not raining, then I will not bring an umbrella.

Converse: If bring an umbrella, then it is raining.

Contrapositive: If I do not bring an umbrella, then it is not raining.

---

p	q	$\sim p$	$\sim q$	Implication $p \Rightarrow q$	Inverse $\sim p \Rightarrow \sim q$	Converse $q \Rightarrow p$	Contrapositive $\sim q \Rightarrow \sim p$

The truth table shows that only the ____________ is ___________________ to the original implication.

Write the contrapositive of each statement:

a) If she has the courage, then she will win.

b) If he studies, then he will pass the exam.

c) If a figure is a rectangle, then it is a parallelogram.

IB Practice A

Consider the following logic statements:

p: x is a factor of 6
q: x is a factor of 24

a) Write $p \Rightarrow q$ in words.

b) Write the converse of $p \Rightarrow q$ in words and in symbolic logic.

c) State whether the converse is true or false.
Give an example to justify your answer.

IB Practice B

Let p and q be the statements: p: Sarah eats lots of carrots.
q: Sarah can see well in the dark.

a) Write the following statements in words.

i) $p \Rightarrow q$

ii) $\sim p \wedge q$

b) Write the following statement in symbolic form:
If Sarah cannot see well in the dark, then she does not eat lots of carrots.

c) Is the statement in part b) the inverse, the converse, or the contrapositive of the statement $p \Rightarrow q$?

IB Practice C

Consider the true statement *'If a figure is a square, then it is a rhombus.'*

a) For this statement, write in words

i) its converse

ii) its inverse

iii) its contrapositive

b) Only one of the statements in part a) is true. Which one is it?

Let p and q be the statements p: You watch the music TV channel.
q: You like music.

a) Consider the following logic statement.

If you watch the music TV channel, then you like music.

i) Write down in words the inverse of the statement.

ii) Write down in words the converse of the statement.

b) Complete the following truth table:

p	q	$\sim p$	$\sim q$	$p \Rightarrow q$	$\sim p \Rightarrow \sim q$	$p \vee \sim q$	$\sim p \wedge q$
T	T						
T	F						
F	T						
F	F						

c) Which of the compound statements in part b) are logically equivalent?

IB Math Studies
Logical Equivalence, Tautologies, and Contradictions

We have already looked at the idea of logical equivalence: an implication and its contrapositive are logically equivalent. However, that is not the only situation where this is found.

If two statements have the same truth value under all circumstances, then the two statements are logically equivalent.

Show that $\sim(q \wedge p)$ is logically equivalent to $\sim q \vee \sim p$.

p	q	$\sim p$	$\sim q$	$q \wedge p$	$\sim(q \wedge p)$	$\sim q \vee \sim p$
T	T					
T	F					
F	T					
F	F					

Show that $\sim(q \vee p)$ is logically equivalent to $\sim q \wedge \sim p$.

p	q	$\sim p$	$\sim q$	$q \vee p$	$\sim(q \vee p)$	$\sim q \wedge \sim p$
T	T					
T	F					
F	T					
F	F					

Show that the inverse and the converse of an implication are logically equivalent.

p	q	$\sim p$	$\sim q$	Implication $p \Rightarrow q$	Inverse	Converse
T	T					
T	F					
F	T					
F	F					

A statement that is always true is a <u>tautology</u>.

A statement that is always false is a <u>contradiction</u>.

Example

p: It is raining.

"It is raining, or it is not raining" is a tautology.

p	$\sim p$	$p \vee \sim p$
T		
F		

"It is raining, and it is not raining" is a contradiction.

p	$\sim p$	$p \wedge \sim p$
T		
F		

Describe the statement $(\sim q \wedge \sim p) \wedge (q \vee p)$.

p	q	$\sim p$	$\sim q$	$\sim q \wedge \sim p$	$q \vee p$	$(\sim q \wedge \sim p) \wedge (q \vee p)$
T	T					
T	F					
F	T					
F	F					

Describe the statement $(q \wedge \sim q) \Rightarrow p$

p	q	$\sim q$	$q \wedge \sim q$	$(q \wedge \sim q) \Rightarrow p$
T	T			
T	F			
F	T			
F	F			