Unit 5 – Exponential/Logarithmic Functions
Exponential Functions (Unit 5.1)

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Lesson Goals

When you have completed this lesson you will:

- Recognize and evaluate exponential functions of base a and base e.
- Apply transformations to exponential functions and graph exponential functions of any base.
- Apply exponential functions to real-life situations.

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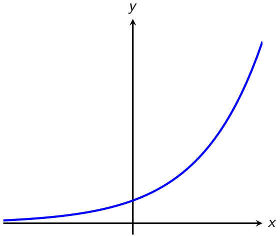
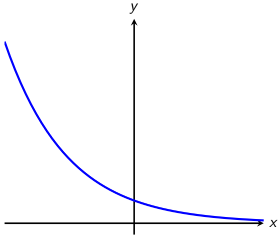
Exponential Functions

A base-b exponential function has the form

$$f(x) = ab^x$$

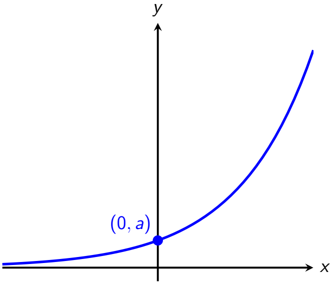
where $a \neq 0$, $b > 0$ and $b \neq 1$, and x is any real number.





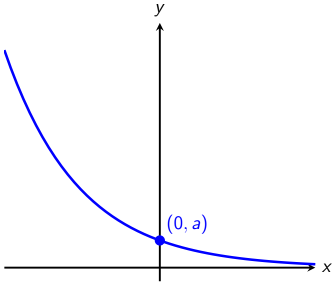
Exponential Growth: $f(x) = ab^x$, $b > 1$

Domain	$(-\infty, \infty)$
Range	$(0, \infty)$
y-Intercept	$(0, a)$
x-Intercept	none
Increasing	$(-\infty, \infty)$
Hor Asymptote	x-axis
Continuous	$(-\infty, \infty)$
End Behavior	$\lim_{x\to-\infty} f(x) = 0$
	$\lim_{x\to\infty} f(x) = \infty$



Exponential Decay: $f(x) = ab^x$, $0 < b < 1$

Domain	$(-\infty, \infty)$
Range	$(0, \infty)$
y-Intercept	$(0, a)$
x-Intercept	none
Decreasing	$(-\infty, \infty)$
Hor Asymptote	x-axis
Continuous	$(-\infty, \infty)$
End Behavior	$\lim_{x \to -\infty} f(x) = \infty$
	$\lim_{x \to \infty} f(x) = 0$



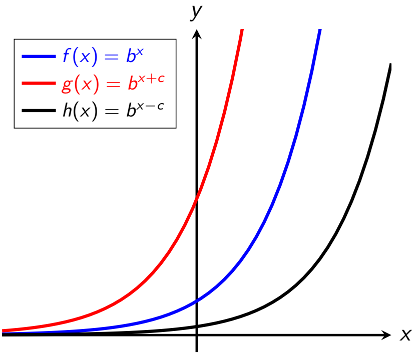
Example 1

Sketch the graph of $f(x) = 2^x$. Then describe the following:

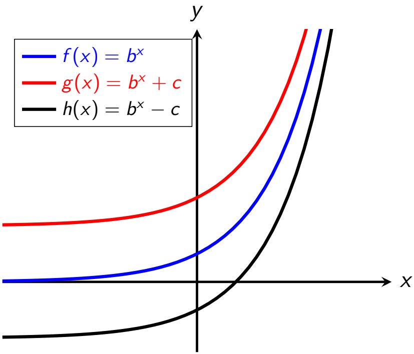
1. Domain
2. Range
3. Intercept
4. Asymptote
5. End behavior
6. Interval incr/decr

Translations

Horizontal Translation c Units

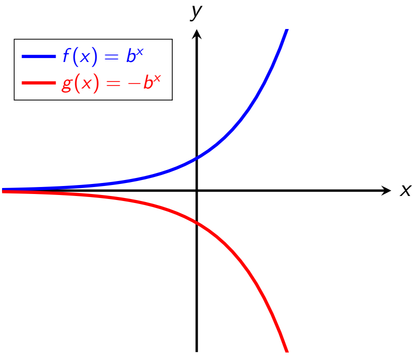


Vertical Translation c Units

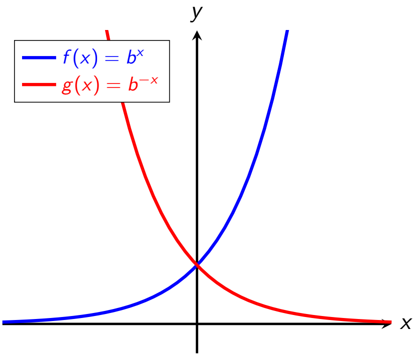


Reflections

Reflect wrt x-axis

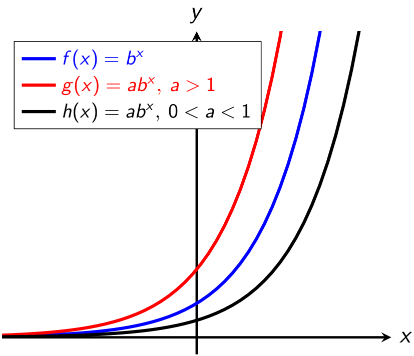


Reflect wrt y-axis

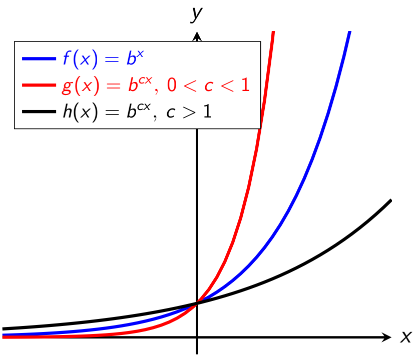


Dilations

Vertical Stretch/Shrink



Horizontal Stretch/Shrink



Example 2

Use the graph of f to describe the transformation that produces the graph of g.

1. $f(x) = 4^x\,, \quad g(x) = 4^{x+2}$

2. $f(x) = 3^x\,, \quad g(x) = 2\left(3^x\right) - 1$

3. $f(x) = 2^x\,, \quad g(x) = 2^{-x}$

The Natural Base e

- Named for the Swiss mathematician Leonhard Euler (pronounced “Oiler”).
- Also called the **natural base**.
- Irrational number $e \approx 2.71828\ldots$.

Graph $y_1 = \left(1 + \frac{1}{x}\right)^x$ and $y_2 = e$ on the same set of axes.
What happens the graph of y_1 as x increases?

Note that e can be found two places on the calculator keyboard: (1) the second function of the division key, and (2) the second function of the LN key.

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Graph of $f(x) = e^x$

Domain	$(-\infty, \infty)$
Range	$(0, \infty)$
y-Intercept	$(0, 1)$
x-Intercept	none
Increasing	$(-\infty, \infty)$
Hor Asymptote	x-axis
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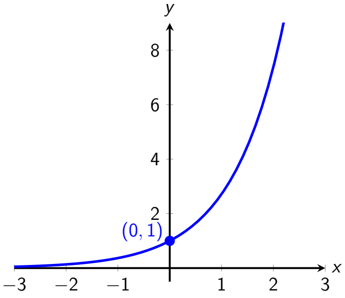

Example 3

Use a calculator to evaluate $f(x) = e^x$ for the indicated values of x.

1. $f(-3)$

2. $f(2)$

Example 4

Use the graph of $f(x) = e^x$ to describe the transformation that results in the graph of each function.

1. $g(x) = e^{4x}$

2. $h(x) = -e^x + 3$

3. $j(x) = \frac{1}{2}e^x$

Compound Interest

Suppose an initial amount of money (called the principal) P is invested at an annual interest rate r and **compounded** once a year. This means that at the end of the first year the earned interest is added to the principal creating a new balance A_1:

$$A_0 = P$$

$$A_1 = A_0 + A_0 r$$

$$A_1 = A_0(1 + r)$$

$$A_1 = P(1 + r)$$

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Compound Interest

Notice the pattern of multiplying the principal by $1 + r$ each year.

Year	Balance After Compounding
0	$A_0 = P$
1	$A_1 = P(1 + r)$
2	$A_2 = A_1(1 + r) = P(1 + r)(1 + r) = P(1 + r)^2$
3	$A_3 = A_2(1 + r) = P(1 + r)^2(1 + r) = P(1 + r)^3$
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t	$A_t = P(1 + r)^t$

Compound Interest

The pattern of annual compounding can be modified to accommodate compounding more often (such as quarterly or monthly, for example) to produce the following **Formula for Compound Interest**:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where

- t is time in years
- $A(t)$ is the total amount after t years
- P is the initial principal
- r is the interest rate
- n is the number of times per year the interest is compounded

Example 5

Suppose you get a summer internship that allows you to save \$5000 to be invested in an interest-bearing account. Calculate how much money you will have if you invest your money for 10 years at 7% and:

1. compound semiannually
2. compound quarterly
3. compound monthly
4. compound daily

Compound Continuously

$$A(t) = P\left(1+\frac{r}{n}\right)^{nt} \qquad \text{(Compound Interest)}$$

$$A(t) = P\left(1+\frac{1}{\frac{n}{r}}\right)^{nt} \qquad \left(\frac{r}{n} = \frac{1}{n/r}\right)$$

$$A(t) = P\left(1+\frac{1}{x}\right)^{(xr)t} \qquad \left(\frac{n}{r} = x \text{ and } n = xr\right)$$

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Compound Continuously

$$A(t) = P\left(1 + \frac{1}{x}\right)^{xrt} \qquad \left(x = \tfrac{n}{r} \text{ and } n = xr\right)$$

$$A(t) = P\left[\left(1 + \frac{1}{x}\right)^{x}\right]^{rt} \qquad \text{(Power Prop Exp)}$$

$$A(t) = P\left[\left(1 + \frac{1}{x}\right)^{x}\right]^{rt} \qquad \left(\lim_{x\to\infty}(1 + 1/x)^x = e\right)$$

$$A(t) = Pe^{rt} \qquad \text{(Substitution)}$$

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Compound Continuously

If you increase the number of times you compound without bound (this means to infinity ...and beyond!) the **Formula for Compound Interest** becomes

$$A(t) = Pe^{rt}$$

where

- t = time in years
- $A(t)$ is the total amount after t years
- P is the initial principal
- r is the interest rate
- e is the natural base

Example 6

Suppose you get a summer internship that allows you to save \$5000 to be invested in an interest-bearing account. Calculate how much money you will have if you invest your money for 10 years at 7% if the interest could be calculated continuously.

Example 7

A population is declining at a rate of 2.5% annually. The current population is approximately 11 million people. Assuming the population decline continues at this rate, predict the population after:

1. 30 years annual decay.
2. 30 years continuous decay.

Example 8

The table below shows the population growth of deer in a forest from 2000 to 2010.

Deer Population

Year	**Deer**
2000	125
2010	264

Assume an exponential rate of growth:

1. Identify the rate of increase.
2. Write an exponential equation to model this situation.
3. Predict the number of deer in 2020.

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- Do problems Chap 3.1 #1, 3, 11-19 odd, 21, 25, 29, 31, 35, 37, 41, 43

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