Name ______________________ Date ______________

BENCHMARK 5
(Chapters 10, 11 and 12)

A. Permutations and Combinations (pp. 96–99)

Making an organized list or using a tree diagram are just two of the methods that can help count the number of ways to perform a task. Other methods are described in the following examples.

1. Use the Fundamental Counting Principle

Vocabulary **Fundamental counting principle** The total number of ways, a_1 through a_n, that n events, can occur is $a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n$.

EXAMPLE **Students using the school computer lab are assigned a password. Each password begins with a letter and is followed by three numbers using the digits 0 to 9.**

a. How many different passwords are possible if the numbers can be repeated?

b. How many different passwords are possible if the numbers cannot be repeated?

Solution:

a. There are 26 choices for each letter and 10 choices for each number. Use the fundamental counting principle to find the number of possible passwords.

$$\text{Number of passwords} = 26 \cdot 10 \cdot 10 \cdot 10 = 26{,}000$$

With repetition, the number of possible passwords is 26,000.

b. If the numbers cannot be repeated, there are still 10 choices for the first number, but only 9 for the second number, and 8 for the third number. Use the fundamental counting principle to find the number of possible passwords.

$$\text{Number of passwords} = 26 \cdot 10 \cdot 9 \cdot 8 = 18{,}720$$

Without repetition, the number of possible passwords is 18,720.

PRACTICE **For extra security, the passwords for the school's computer lab are changed so that each password contains two letters followed by 4 digits.**

1. How many different passwords are possible if the letters and digits can be repeated?

2. How many different passwords are possible if both letters and digits cannot be repeated?

Name ______________________________ Date ______________

BENCHMARK 5

(Chapters 10, 11 and 12)

2. Find Permutations

Vocabulary

Permutation An ordering of n objects.

Factorial The product of all integers from the specified integer down to one.

Factorial symbol !, with $n!$ read as "n factorial".

Zero factorial is defined as $0! = 1$.

EXAMPLE

a. There are 12 students competing at a science fair. In how many different ways can 3 of the students win first, second, and third place prizes?

The number of permutations of n distinct objects is $n!$.

b. From the 11 people running for office, one will be chosen president, another will be chosen vice-president, a third will be chosen treasurer, and a fourth will be chosen secretary. In how many different ways can these offices be filled?

c. Find the number of distinguishable permutations of the letters in the word POPCORN.

The number of permutations of n objects taken r at a time is given by the formula:

$${}_nP_r = \frac{n!}{(n-r)!}$$

The number of distinguishable permutations of n objects where one object is repeated s_1 times, another is repeated s_2 times, and so on, is:

$$\frac{n!}{s_1! \cdot s_2! \cdot \ldots \cdot s_k!}$$

Solution:

a. Any of the 12 students can win the first place prize, then any of the remaining 11 students can win the second place prize, and finally, any of the remaining 10 students can win the third place prize. So, the number of different ways the students can win a prize is:

$$12 \cdot 11 \cdot 10 = 1320$$

b. Find the number of permutations of 11 people taken 4 at a time.

$${}_{11}P_4 = \frac{11!}{(11-4)!} = \frac{11!}{7!} = \frac{39{,}916{,}800}{5040} = 7920$$

c. POPCORN has 7 letters of which P and O are each repeated 2 times. So, the number of distinguishable permutations is:

$$\frac{7!}{2! \cdot 2!} = \frac{5040}{2 \cdot 2} = 1260$$

PRACTICE

3. At a track meet, 8 people are running in the 100-meter dash. Assuming there are no ties, in how many ways can 4 of the runners finish in first through fourth place?

4. Find the number of distinguishable permutations of the letters in the word AMERICAN.

Name ______________________ Date ______________

BENCHMARK 5

(Chapters 10, 11 and 12)

3. Find Combinations

Vocabulary

Combination A selection of r objects from a group of n distinct objects where the order is not important.

EXAMPLE

The formula for the number of combinations of n objects taken r at a time is:

$${}_nC_r = \frac{n!}{(n-r)! \cdot r!}$$

The coach chooses 2 of the 10 members on the volleyball team to be captains. In how many ways can 2 of the 10 members be chosen?

Solution:

The order the 2 captains are chosen is not important. Find the number of combinations of 10 people taken 2 at a time.

$${}_{10}C_2 = \frac{10!}{(10-2)! \cdot 2!} = \frac{10!}{8! \cdot 2!} = \frac{3{,}628{,}800}{80{,}640} = 45$$

PRACTICE

5. For a game, the names of 8 movies are put into a hat. Of those, 4 are randomly chosen. In how many different ways can 4 movies be chosen from the 8 movies named?

4. Use Pascal's Triangle

Vocabulary

Pascal's triangle A triangular pattern showing the values of ${}_nC_r$ such that each row corresponds to a value of n.

$n = 0$ (0th row) ${}_0C_0$

$n = 1$ (1st row) ${}_1C_0$ ${}_1C_1$

$n = 2$ (2nd row) ${}_2C_0$ ${}_2C_1$ ${}_2C_2$

$n = 3$ (3rd row) ${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$

$n = 4$ (4th row) ${}_4C_0$ ${}_4C_1$ ${}_4C_2$ ${}_4C_3$ ${}_4C_4$

$n = 5$ (5th row) ${}_5C_0$ ${}_5C_1$ ${}_5C_2$ ${}_5C_3$ ${}_5C_4$ ${}_5C_5$

EXAMPLE

Of the 7 seniors in the drama club, 3 will be chosen to sell tickets to the school play. Use Pascal's triangle to find the number of combinations of 3 seniors who can be chosen to sell tickets.

Solution:

You need to find ${}_7C_3$. Write the 6th and 7th rows of Pascal's triangle by adding numbers from the previous row.

$n = 5$ (5th row) 1 5 10 10 5 1

$n = 6$ (6th row) 1 6 15 20 15 6 1

$n = 7$ (7th row) 1 7 21 35 35 21 7 1

${}_7C_0$ ${}_7C_1$ ${}_7C_2$ ${}_7C_3$ ${}_7C_4$ ${}_7C_5$ ${}_7C_6$ ${}_7C_7$

As shown above, the value of ${}_7C_3$ is the fourth number in the 7th row of Pascal's triangle. So, ${}_7C_3 = 35$, indicating there are 35 ways to choose 3 seniors from the drama club to sell tickets.

PRACTICE

6. Use Pascal's triangle to find the number of combinations of 2 juniors who can be chosen to sell tickets if there are 8 juniors in the drama club.

Name ______________________ Date ______________

BENCHMARK 5

(Chapters 10, 11 and 12)

5. Use the Binomial Theorem

Vocabulary

Binomial theorem A generalization of a binomial expansion $(a + b)^n$, where n is a positive integer, given by:

$$(a + b)^n = {}_nC_0 a^n b^0 + {}_nC_1 a^{n-1} b^1 + {}_nC_2 a^{n-2} b^2 + \ldots + {}_nC_n a^0 b^n$$

EXAMPLE

The numbers in Pascal's triangle can be used to find coefficients in binomial expansions.

When the coefficients in the binomial do not equal 1, the coefficients in the binomial expansion are not the same as the corresponding row in Pascal's triangle.

Use the binomial theorem to write the binomial expansion.

a. $(m + n^2)^3$ **b.** $(3p - q)^4$

Solution:

a. $(m + n^2)^3 = {}_3C_0 m^3(n^2)^0 + {}_3C_1 m^2(n^2)^1 + {}_3C_2 m^1(n^2)^2 + {}_3C_3 m^0(n^2)^3$

$= (1)m^3(1) + (3)m^2n^2 + (3)mn^4 + (1)(1)n^6$

$= m^3 + 3m^2n^2 + 3mn^4 + n^6$

b. $(3p - q)^4 = [3p + (-q)]^4$

$= {}_4C_0(3p)^4(-q)^0 + {}_4C_1(3p)^3(-q)^1 + {}_4C_2(3p)^2(-q)^2 + {}_4C_3(3p)^1(-q)^3 + {}_4C_4(3p)^0(-q)^4$

$= 1(81p^4)(1) + 4(27p^3)(-q) + 6(9p^2)(q^2) + 4(3p)(-q^3) + 1(1)(q^4)$

$= 81p^4 - 108p^3q + 54p^2q^2 - 12pq^3 + q^4$

PRACTICE

Use the binomial theorem to write the binomial expansion.

7. $(x - y)^5$ **8.** $(a^2 + 4b)^3$

Quiz

Determine the number of possible passwords with repeated letters and digits.

1. 3 letters followed by 2 digits **2.** 4 digits followed by 2 letters

Determine the number of possible passwords without repeated letters and digits.

3. 2 digits followed by 4 letters **4.** 3 letters followed by 3 digits

Find the number of distinguishable permutations of the letters in the word.

5. MASSIVE **6.** COMMON **7.** YOYO

Find the number of permutations or combinations.

8. ${}_7P_3$ **9.** ${}_6P_3$ **10.** ${}_{12}C_7$

11. ${}_4C_1$ **12.** ${}_5P_4$ **13.** ${}_7C_5$

Write the binomial expansion.

14. $(d + 3)^4$ **15.** $(2v - 3w)^3$ **16.** $(r + s^2)^5$

Name ______________________ Date ____________

BENCHMARK 5

(Chapters 10, 11 and 12)

B. Probability (pp. 100–103)

In games of chance, the possible outcomes or collection of outcomes are known as *events*. Numbers between 0 and 1 represent the likelihood of events occurring. How to find these likelihoods is described below.

1. Find a Theoretical Probability and Odds

Vocabulary

Probability A number between 0 and 1 that indicates the likelihood of an event occurring.

Theoretical probability The number of possible outcomes in a specific event divided by the total number of outcomes, when all outcomes are equally likely.

Odds A measure of the chances in *favor* of an event occurring or of chances *against* an event occurring, when all outcomes are equally likely.

EXAMPLE **You roll a standard six-sided die.**

a. What is the probability of rolling an odd number?

b. Find the odds in favor of rolling a number greater than 4.

Odds in *favor* of event A = (number of outcomes in A) ÷ (number of outcomes not in A)

Odds *against* event A = (number of outcomes not in A) ÷ (number of outcomes in A)

Solution:

a. There are 6 possible outcomes. A total of 3 outcomes correspond to rolling an odd number: 1, 3, and 5.

$$P(\text{rolling odd number}) = \frac{\text{number of ways to roll an odd number}}{\text{number of ways to roll the die}} = \frac{3}{6} = \frac{1}{2}$$

b. Odds in favor of rolling number greater than 4 $= \dfrac{\text{numbers greater than 4}}{\text{numbers less than or equal to 4}}$

Two numbers are greater than 4: 5 and 6. Four numbers are less than or equal to 4: 1, 2, 3, and 4.

Odds in favor of rolling number greater than 4 $= \dfrac{2}{4} = \dfrac{1}{2}$

PRACTICE **A card is randomly drawn from a standard deck of 52 cards. Find the probability and odds in favor of drawing the given card.**

1. a queen of diamonds

2. a red card

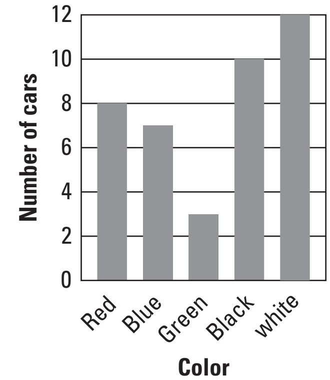
3. a black ace

BENCHMARK 5
(Chapters 10, 11 and 12)

2. Find an Experimental Probability

Vocabulary

Experimental probability The number of trials where a specific event occurs divided by the total number of trials in an experiment.



EXAMPLE

Probabilities can be written as fractions, decimals, or percents between 0 and 100.

The graph shows the color of each car in a car dealership's parking lot. Find the probability that a car randomly selected for a test drive would be black.

Solution:

The total number of cars in the parking lot is $8 + 7 + 3 + 10 + 12 = 40$.

Of those, 10 are black. So, $P(\text{black}) = \frac{10}{40} = \frac{1}{4}$.

PRACTICE

4. What is the experimental probability that a car randomly selected for a test drive is either red or blue?

5. What is the experimental probability that a car randomly selected for a test drive is *not* white?

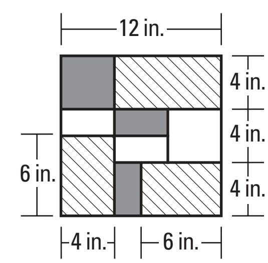
3. Find a Geometric Probability

Vocabulary

Geometric probability A probability found by calculating a ratio of two lengths, areas, or volumes.

EXAMPLE

A board used in a game has different sized white, black, and striped rectangles. A coin is tossed and lands in one of the rectangles. What is the probability that the coin lands in a striped rectangle?



Solution:

$$P(\text{striped}) = \frac{\text{Area of striped rectangles}}{\text{Area of entire board}} = \frac{8(4) + 4(6) + 6(4)}{12(12)} = \frac{80}{144} = \frac{5}{9}$$

PRACTICE

6. What is the probability of the coin landing in a white square?

Name ______________________ Date ______________

BENCHMARK 5

(Chapters 10, 11 and 12)

4. Find Probabilities of Disjoint or Overlapping Events

Vocabulary

Compound event The union or intersection of two events.

Overlapping events Two events that have one or more outcomes in common.

Disjoint or mutually exclusive events Two events that have no outcomes in common.

EXAMPLE

If A and B are any two events, then $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

If A and B are disjoint events, then $P(A \text{ or } B) = P(A) + P(B)$.

A card is randomly selected from a standard deck of 52 playing cards.

a. What is the probability that the card is a red eight *or* an ace?

b. What is the probability that the card is a face card *or* a diamond?

Solution:

a. Let event A be selecting a red eight and event B be selecting an ace. A has 2 outcomes and B has 4 outcomes. Because A and B are disjoint, the probability is:

$$P(A \text{ or } B) = P(A) + P(B) = \frac{2}{52} + \frac{4}{52} = \frac{6}{52} = \frac{3}{26} \approx 0.115$$

b. Let event A be selecting a face card and event B be selecting a diamond. A has 12 outcomes and B has 13 outcomes. Of these, 3 outcomes are common to both A and B. The probability of selecting a face card or a diamond is:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = \frac{12}{52} + \frac{13}{52} - \frac{3}{52} = \frac{22}{52} = \frac{11}{26} \approx 0.423$$

PRACTICE

A card is randomly selected from a standard deck of 52 playing cards. Find the probability of the given event.

7. selecting a six *or* a heart

8. selecting a 10 *or* a jack

5. Find a Conditional Probability

Vocabulary

The conditional probability of B given A is written as $P(B|A)$.

Dependent events Two events, A and B, in which the occurrence of one event affects the occurrence of the other event.

Conditional probability The probability that one event will occur given that another has occurred.

EXAMPLE

If A and B are dependent events, the probability both events will occur is: $P(A \text{ and } B) = P(A) \cdot P(B|A)$.

Two cards are randomly selected from a standard deck of 52 cards. The first card is *not* replaced. Find the probability that the first card is a red 5 and the second card is black.

Solution:

Let A be "the first card is a red 5" and B be "the second card is black." A and B are dependent events since the first card is not replaced before selecting the second card. So, the probability is:

$$P(A \text{ and } B) = P(A) \cdot P(B|A) = \frac{2}{52} \cdot \frac{26}{51} = \frac{1}{51} \approx 0.020$$

PRACTICE

Find the probability of drawing the given cards from a standard deck of 52 cards without replacement.

9. a heart, then a spade

10. a face card, then a red ace

Name ____________________ Date ____________

BENCHMARK 5
(Chapters 10, 11 and 12)

6. Find the Probability of Independent and Dependent Events

Vocabulary

Independent events Two events, A and B, in which the occurrence of one event has no effect on the occurrence of the other event.

EXAMPLE

If A and B are independent events, the probability both events will occur is: $P(A \text{ and } B) = P(A) \cdot P(B)$.

Two shapes are randomly chosen from a bag containing 6 squares, 15 circles, and 12 triangles. Find the probability of selecting two circles if (a) the first shape selected is replaced and if (b) the first shape selected is not replaced.

Solution:

a. $P(A \text{ and } B) = P(A) \cdot P(B) = \frac{15}{33} \cdot \frac{15}{33} = \frac{25}{121} \approx 0.207$

b. $P(A \text{ and } B) = P(A) \cdot P(B|A) = \frac{15}{33} \cdot \frac{14}{32} = \frac{35}{176} \approx 0.199$

PRACTICE

Find the probability of drawing the given cards from a standard deck of 52 cards (a) with replacement and (b) without replacement.

11. a jack, then an ace

12. two face cards

Quiz

A standard six-sided die is rolled. Find the probability of rolling each number.

1. a two

2. a multiple of three

3. a number less than six

A card is randomly drawn from a standard deck of 52 cards. Find the odds in favor of drawing the given card.

4. a seven of clubs

5. a red three or four

6. not an ace

A bag contains 6 yellow chips, 4 blue chips, and 5 white chips. Find the probability of selecting these chips.

7. a white chip

8. a white or a blue chip

9. not a blue chip

A card is randomly selected from a standard deck of 52 playing cards. Find the probability of the given event.

10. selecting a red five *or* a black card

11. selecting a red face card *or* a queen

Find the probability of drawing the given cards from a standard deck of 52 cards (a) with replacement and (b) without replacement.

12. a black three, then a nine

13. a seven, then another seven

Name ______________________ Date ______________

BENCHMARK 5

(Chapters 10, 11 and 12)

C. Probability Distributions (pp. 104–106)

In some experiments, the probabilities of all possible outcomes can be collected and studied. The concept of these probability collections, or distributions, is explored further here.

1. Use a Binomial Distribution

Vocabulary

The sum of all probabilities in a probability distribution must equal 1.

Probability distribution A function that gives the probability of each possible value of a random variable.

Binomial experiment An experiment meeting the following conditions:

- There are n independent trials.
- Each trial has two possible outcomes: success and failure.
- The probability of success for each trial, p, is the same. The probability of failure is denoted by $1 - p$.

Binomial distribution A type of probability distribution that shows the outcomes of a binomial experiment.

EXAMPLE **According to a survey at one high school, 64% of students have a part-time job. Five randomly chosen students are asked whether they have a part-time job.**

a. Draw a histogram of the binomial distribution for the survey.

b. Use the binomial distribution to determine the probability that at least 3 out of the 5 randomly chosen students have a part-time job.

Solution:

The probability of exactly k successes in n trials of a binomial experiment is: $P(k \text{ successes}) = {}_nC_k p^k(1 - p)^{n-k}$

a. The probability that a randomly selected student has a part-time job is $p = 0.64$. Since 5 students are surveyed, $n = 5$.

$P(k = 0) = {}_5C_0(0.64)^0(0.36)^5 \approx 0.006$

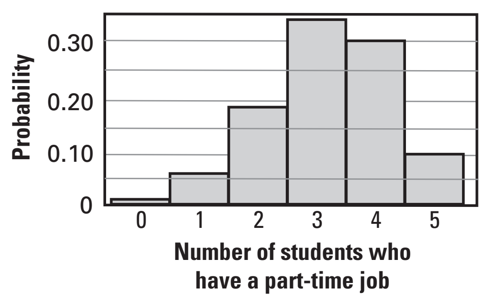
$P(k = 1) = {}_5C_1(0.64)^1(0.36)^4 \approx 0.054$

$P(k = 2) = {}_5C_2(0.64)^2(0.36)^3 \approx 0.191$

$P(k = 3) = {}_5C_3(0.64)^3(0.36)^2 \approx 0.340$

$P(k = 4) = {}_5C_4(0.64)^4(0.36)^1 \approx 0.302$

$P(k = 5) = {}_5C_5(0.64)^5(0.36)^0 \approx 0.107$



b. The probability at least 3 out of the 5 students have a part-time job is $0.34 + 0.302 + 0.107 = 0.749$.

PRACTICE **The top-scoring basketball player on the team made 80% of his attempted free-throws.**

1. Draw a histogram of the binomial distribution showing the expected number of free-throws this player makes during his next 4 attempts.

2. What is the probability the basketball player makes at least 3 out of 4 attempted free-throws?

Name ______________________ Date ______________

BENCHMARK 5

(Chapters 10, 11 and 12)

2. Find a Normal Probability

Vocabulary

The total area under the normal curve is 1.

Mean The sum of n numbers divided by n.

Standard deviation The typical difference between a data value and the mean of a data set.

Normal distribution A type of probability distribution modeled by a bell-shaped curve.

Normal curve The bell-shaped curve in a normal distribution that is symmetric about the mean.

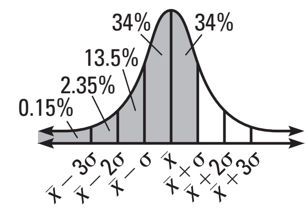
EXAMPLE

About 68% of the area under the normal curve lies within 1 standard deviation from the mean; about 95% lies within 2 standard deviations of the mean; about 99.7% lies within 3 standard deviations of the mean.

A normal distribution has a mean of $\bar{x}$ and a standard deviation σ. For a randomly selected *x*-value from the distribution, find $P(x \le \bar{x} + \sigma)$.

Solution:

The probability that a randomly selected x-value lies below $\bar{x} + \sigma$ is shaded under the normal curve shown. Therefore, $P(x \le \bar{x} + \sigma) =$ $0.34 + 0.34 + 0.135 + 0.0235 + 0.0015 = 0.84.$



PRACTICE

A normal distribution has a mean of $\bar{x}$ and a standard deviation σ. Find the indicated probability for a randomly selected *x*-value from the distribution.

3. $P(x \ge \bar{x} + 2\sigma)$ **4.** $P(x \le \bar{x} - \sigma)$ **5.** $P(\bar{x} - \sigma \le x \le \bar{x} + 3\sigma)$

3. Use a *z*-score and the Standard Normal Table

Vocabulary

The formula for the *z*-score is: $z = \frac{(x - \bar{x})}{\sigma}$

***z*-score** The z-value for a particular x-value that shows the number of standard deviations the x-value lies above or below the mean, $\bar{x}$.

Standard normal table A table showing the probability that z, a randomly selected value from a normal distribution, is less than or equal to some given value.

Standard Normal Table

z	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
−3	.0013	.0010	.0007	.0005	.0003	.0002	.0002	.0001	.0001	.0000+
−2	.0228	.0179	.0139	.0107	.0082	.0062	.0047	.0035	.0026	.0019
−1	.1587	.1357	.1151	.0968	.0808	.0668	.0548	.0446	.0359	.0287
−0	.5000	.4602	.4207	.3821	.3446	.3085	.2743	.2420	.2119	.1841
0	.5000	.5398	.5793	.6179	.6554	.6915	.7257	.7580	.7881	.8159
1	.8413	.8643	.8849	.9032	.9192	.9332	.9452	.9554	.9641	.9713
2	.9772	.9821	.9861	.9893	.9918	.9938	.9953	.9965	.9974	.9981
3	.9987	.9990	.9993	.9995	.9997	.9998	.9998	.9999	.9999	1.0000–

Name ____________________ Date ____________

BENCHMARK 5

(Chapters 10, 11 and 12)

EXAMPLE **A normal distribution has a mean of 22 and a standard deviation of 5. Use the standard normal table to find the indicated probability.**

a. $P(x \leq 13)$ **b.** $P(25 < x \leq 32)$

Solution:

a. **Step 1: Find** the z-score corresponding to an x-value of 13.

$$z = \frac{x - \bar{x}}{\sigma} = \frac{13 - 22}{5} = -1.8$$

Step 2: Use the table to find $P(x \leq 13) = P(z \leq -1.8)$.

The table shows that $P(z \leq -1.8) = 0.0359$. So, the probability that $x \leq 13$ is 0.0359.

b. **Step 1: Find** the z-score corresponding to x-values of 25 and 32.

$$z = \frac{x - \bar{x}}{\sigma} = \frac{25 - 22}{5} = 0.6 \qquad z = \frac{x - \bar{x}}{\sigma} = \frac{32 - 22}{5} = 2.0$$

Step 2: Use the table to find $P(0.6 < z \leq 2.0)$.

The table shows that $P(z \leq 0.6) = 0.7257$. The table shows that $P(z \leq 2.0) = 0.9772$. Therefore, $P(0.6 < z \leq 2.0)$ is $P(z \leq 2.0) - P(z \leq 0.6) = 0.9772 - 0.7257 = 0.2515$.

PRACTICE **A normal distribution has a mean of 89 and a standard deviation of 11. Use the standard normal table to find the indicated probability.**

6. $P(x \leq 80)$ **7.** $P(62 < x \leq 76)$ **8.** $P(x > 95)$

Quiz

There is a 25% chance it will rain sometime during the next four days.

1. Draw a histogram of the binomial distribution for this forecast.

2. What is the probability that it will rain at most 2 days?

A normal distribution has a mean of $\bar{x}$ and a standard deviation σ. Find the indicated probability for a randomly selected x-value from the distribution.

3. $P(x \leq \bar{x} + 2\sigma)$ **4.** $P(x \geq \bar{x} - 3\sigma)$ **5.** $P(\bar{x} \leq x \leq \bar{x} + \sigma)$

A normal distribution has a mean of 6.3 and a standard deviation of 1.5. Use the standard normal table to find the indicated probability.

6. $P(x \leq 6.8)$ **7.** $P(x > 5.5)$ **8.** $P(7.3 < x \leq 8.0)$

Name ______________________ Date ______________

BENCHMARK 5
(Chapters 10, 11 and 12)

D. Data Analysis (pp. 107–111)

You learned how to analyze data by creating and interpreting graphs and other data displays. Here you will learn how numerical values, known as *statistics*, are used to summarize and compare data sets.

1. Find Measures of Central Tendency

Vocabulary

The mean is denoted by $\bar{x}$.

Measure of central tendency A number used to represent the center or middle of a set of data values.

Mean The sum of n numbers divided by n.

Median The middle number in an ordered list of n numbers when n is odd; the mean of the two middle numbers in an ordered list of n numbers when n is even.

Mode The number or numbers that occur most frequently.

EXAMPLE

A data set may have no mode, one mode, or more than one mode.

The table shows the running times, in minutes, of movies playing at the local theater. Find the mean, median, and mode of this data set.

Movie Times		
108	98	122
94	104	108
88	100	114

Solution:

Mean: $\bar{x} = \frac{108 + 98 + 122 + 94 + 104 + 108 + 88 + 100 + 114}{9} = \frac{936}{9} = 104$

Ordered list: 88, 94, 98, 100, 104, 108, 108, 114, 122

Median: 104

Mode: 108

PRACTICE

The data set below shows the waiting times, in minutes, of patients at a dentist's office.

15, 8, 10, 4, 22, 7, 19, 26, 14, 10

1. Find the mean, median, and mode of the data set.

2. Find Measures of Dispersion

Vocabulary

Measure of dispersion A statistic that tells how *dispersed*, or spread out, data values are.

Range A measure of dispersion that shows the difference between the greatest and least data values in a data set.

Standard deviation Another measure of dispersion that describes the *deviation*, or typical difference, between a data value and the mean of a data set.

Name ______________________ Date ____________

BENCHMARK 5

(Chapters 10, 11 and 12)

EXAMPLE **Find the (a) range and (b) standard deviation in movie running times listed at the right.**

Movie Times (minutes)		
108	98	122
94	104	108
88	100	114

The standard deviation σ of $x_1, \ldots, x_n$ is: $\sqrt{\frac{(x_1 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n}}$

Solution:

a. The longest running time is 122 minutes.
The shortest running time is 88 minutes.

Range: $122 - 88 = 34$ minutes

b. $\bar{x} = 104$, so $\sigma = \sqrt{\frac{(108 - 104)^2 + (98 - 104)^2 + \ldots + (114 - 104)^2}{9}} = \sqrt{\frac{864}{9}} \approx 9.8$

The standard deviation of the running times is about 9.8 minutes.

PRACTICE

2. Find the (a) range and (b) standard deviation of the wait times, in minutes, listed below.

15, 8, 10, 4, 22, 7, 19, 26, 14, 10

3. Apply Transformations to Data

EXAMPLE **The annual salaries of eight employees at a bookstore, in thousands of dollars, are listed below.**

25, 37, 18, 20, 25, 35, 34, 30

a. Find the mean, median, mode, range, and standard deviation of the salaries.

b. Each employee gets an annual salary increase of $2000. Find the mean, median, mode, range, and standard deviation of the salaries including the annual increase.

When a constant is added to each value in a data set, the range and standard deviation are unchanged. The mean, median, and mode increase by the constant amount.

Solution:

a. Mean: $28,000 Median: $27,500 Mode: $25,000
Range: $19,000 Standard Deviation: ≈ $6633

b. Mean: $28,000 + $2000 = $30,000 Median: $27,500 + $2000 = $29,500
Mode: $25,000 + $2000 = $27,000 Range: $19,000
Standard Deviation: ≈ $6633

PRACTICE **Find the mean, median, mode, range, and standard deviation of (a) the given data set and (b) the data set obtained by adding the given constant to each data value.**

3. 277, 382, 272, 382, 173, 296; constant 150

EXAMPLE **The distances traveled in yards by golf balls at a driving range are listed below.**

150, 120, 135, 140, 120, 145, 160, 150

a. Find the mean, median, mode, range, and standard deviation of the distances in yards.

b. Find the mean, median, mode, range, and standard deviation of the distances in feet.

Name ______________________ Date ____________

BENCHMARK 5

(Chapters 10, 11 and 12)

When each value in a data set is multiplied by a constant, the mean, median, mode, range, and standard deviation of the new data are multiplied by the constant.

Solution:

a. Mean: 140 Median: 142.5 Modes: 120 and 150
Range: 40 Standard Deviation: ≈ 13.5

b. There are 3 feet in 1 yard, so the constant multiplier is 3.
Mean: $140 \cdot 3 = 420$ Median: $142.5 \cdot 3 = 427.5$
Modes: $120 \cdot 3 = 360$ and $150 \cdot 3 = 450$ Range: $40 \cdot 3 = 120$
Standard Deviation: $\approx 13.5 \cdot 3 = 40.5$

PRACTICE **Find the mean, median, mode, range, and standard deviation of (a) the given data set and (b) the data set obtained by multiplying each data value by the given constant.**

4. 18, 48, 27, 41, 39, 40, 25; constant 3.2

4. Classify Samples

Vocabulary

Population A group of people or objects you want information about.

Sample A subset of the population.

Self-selected sample A group containing volunteer members of a population.

Systematic sample A sample obtained by a rule used to select members of a population.

Convenience sample A group of easy-to-reach members of a population.

Random sample A group of members of a population that have an equal chance of being selected.

Unbiased sample A representation of the population you want information about.

Biased sample An underrepresented or overrepresented group of the population.

EXAMPLE **The captain of the basketball team wants to survey students in her school to find out their favorite sport. She asks every 20th student she sees enter the school one day. Identify the type of sample described and tell whether the sample is biased.**

Solution:

The captain of the team uses a rule of asking every 20th student. So, the sample is a systematic sample. Since the captain chooses a good representation of students, the sample is not biased.

PRACTICE **Identify the type of sample described, and tell whether the sample is biased.**

5. The captain asks various students she selects from the school's attendance list without looking.

6. The captain asks students she sees in the gym during her basketball practice.

Name ______________________ Date ____________

BENCHMARK 5

(Chapters 10, 11 and 12)

5. Find a Margin of Error

Vocabulary

Margin of error A percentage that gives a limit on how much the responses of a sample would differ from the responses of the population.

EXAMPLE

For a sample of n people, the margin of error is: $\pm\frac{1}{\sqrt{n}}$

In a survey of 175 people, 33% said they prefer to go out to a movie rather than rent a movie.

a. What is the margin of error for the survey?

b. Give an interval that is likely to contain the exact percent of the population who would rather go out to a movie than rent one.

Solution:

a. margin of error $= \pm\frac{1}{\sqrt{175}} \approx \pm 0.076$, so the margin of error is about $\pm 7.6\%$

b. To find the interval, subtract and add 7.6% to the percent of people surveyed who said they prefer to go out to a movie rather than rent one (33%).

$33\% - 7.6\% = 25.4\%$ $\qquad$ $33\% + 7.6\% = 40.6\%$

It is likely that between 25.4% and 40.6% of the population would prefer to go out to a movie rather than rent one.

PRACTICE

7. In the survey described above, the number of people surveyed increases to 280 and the results remain the same. Find the new margin of error and likely interval for the population.

6. Choose the Best Model

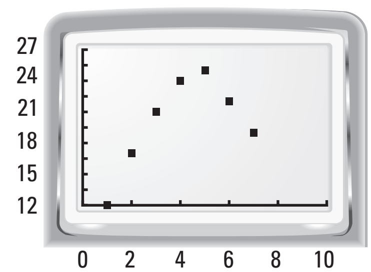
EXAMPLE

Which type of function best models these data points?

x	1	2	3	4	5	6	7
y	12	17	21	24	25	22	19

Solution:

Make a scatter plot. The points form an inverted U-shape. This suggests a quadratic model.



PRACTICE

Which type of function best models these data points?

8.

x	5	10	15	20	25	30	35
y	2	3	5	10	16	24	35

Name ______________________ Date ____________

BENCHMARK 5

(Chapters 10, 11 and 12)

Quiz

The data set below lists the points scored by each player in a ball game during their first home game.

6, 11, 2, 0, 24, 11, 10, 5, 3

1. Find the mean, median, and mode of the data set.
2. Find the range and standard deviation of the data set.

Find the mean, median, mode, range, and standard deviation of (a) the given data set and (b) the data set obtained by adding the given constant to each data value.

3. 28, 39, 98, 37, 27, 48, 63, 28; constant 10

Find the mean, median, mode, range, and standard deviation of (a) the given data set and (b) the data set obtained by multiplying each data value by the given constant.

4. 3.9, 2.8, 3.7, 5.1, 6.4, 3.3; constant 5

A survey is conducted to determine how many books students read each year for pleasure. Identify the type of sample described, and tell whether the sample is biased.

5. Every other student that enters the school library is asked to take the survey.
6. Students in the school's cafeteria are asked throughout the day whether they want to participate in the survey.

In a survey of 930 people, 26% said they grocery shop more than once a week.

7. What is the margin of error for the survey?
8. Give an interval that is likely to contain the exact percent of the population who grocery shop more than once a week.

Which type of function best models these data points?

9.

x	0	1	2	3	4	5	6
y	21.6	19.1	16.8	14.5	11.9	9.0	6.8

Name ______________________ Date ______________

BENCHMARK 5
(Chapters 10, 11 and 12)

E. Sequences and Series (pp. 112–117)

Number patterns can be described using certain rules, or equations. The skills you learned identifying and writing functions will help you write the rules for various number patterns.

1. Write Terms of Sequences

Vocabulary

Sequence A function whose domain is the set of positive integers or the first n positive integers.

Terms of a sequence The values in the range of a function.

EXAMPLE **Write the first six terms of the sequence.**

A *finite* sequence has a limited number of terms.

An *infinite* sequence continues with no end.

a. $f(n) = 4n - 1$ **b.** $a_n = 2^{n+1}$

Solution:

a.	$f(1) = 4(1) - 1 = 3$	**1st term**	**b.** $a_1 = 2^{1+1} = 4$	**1st term**
	$f(2) = 4(2) - 1 = 7$	**2nd term**	$a_2 = 2^{2+1} = 8$	**2nd term**
	$f(3) = 4(3) - 1 = 11$	**3rd term**	$a_3 = 2^{3+1} = 16$	**3rd term**
	$f(4) = 4(4) - 1 = 15$	**4th term**	$a_4 = 2^{4+1} = 32$	**4th term**
	$f(5) = 4(5) - 1 = 19$	**5th term**	$a_5 = 2^{5+1} = 64$	**5th term**
	$f(6) = 4(6) - 1 = 23$	**6th term**	$a_6 = 2^{6+1} = 128$	**6th term**

PRACTICE **Write the first six terms of the sequence.**

1. $f(n) = \frac{n+2}{n}$ **2.** $a_n = 2 - 3n$ **3.** $a_n = (-3)^{2n}$

2. Find the Sum of a Series

Vocabulary

Series The sum of the terms of a sequence, either finite or infinite.

EXAMPLE **Find the sum of the series $\sum_{i=3}^{7}(i+3)^2$.**

Series are summed using summation notation, also known as sigma notation. For example:

$1 + 2 + 3 = \sum_{i=1}^{3} i$

$1 + 2 + 3 + \ldots = \sum_{i=1}^{\infty} i$

Solution:

$$\sum_{i=3}^{7}(i+3)^2 = (3+3)^2 + (4+3)^2 + (5+3)^2 + (6+3)^2 + (7+3)^2$$
$$= 36 + 49 + 64 + 81 + 100 = 330$$

PRACTICE **Find the sum of the series.**

4. $\sum_{i=1}^{5} 3 - 2i$ **5.** $\sum_{n=1}^{8} 2n$ **6.** $\sum_{k=8}^{12}(k-6)^2$

Name ______________________ Date ______________

BENCHMARK 5
(Chapters 10, 11 and 12)

3. Analyze Arithmetic Sequences and Series

Vocabulary

Arithmetic sequence A sequence where the difference between consecutive terms is constant.

Common difference The constant difference between terms in an arithmetic sequence, denoted by d.

EXAMPLE **Write a rule for the nth term of the sequence. Then find a_{21}.**

a. 3, 9, 15, 21, … **b.** 216, 210, 204, 198, …

Solution:

The rule for the nth term, a_n, in an arithmetic sequence is given by: $a_n = a_1 + (n - 1)d$

a. The sequence is arithmetic with first term $a_1 = 3$ and common difference $d = 9 - 3 = 6$. So the rule for the nth term is:

$$\begin{aligned} a_n &= a_1 + (n-1)d && \textbf{Write general rule.} \\ &= 3 + (n-1)(6) && \textbf{Substitute 3 for } a_1 \textbf{ and 6 for } d. \\ &= -3 + 6n && \textbf{Simplify.} \end{aligned}$$

The 21st term is $a_{21} = -3 + 6(21) = 123$.

b. The sequence is arithmetic with first term $a_1 = 216$ and common difference $d = 210 - 216 = -6$. So the rule for the nth term is:

$$\begin{aligned} a_n &= a_1 + (n-1)d && \textbf{Write general rule.} \\ &= 216 + (n-1)(-6) && \textbf{Substitute 216 for } a_1 \textbf{ and –6 for } d. \\ &= 222 - 6n && \textbf{Simplify.} \end{aligned}$$

The 21st term is $a_{21} = 222 - 6(21) = 96$.

EXAMPLE **Find the sum of the arithmetic series $\sum_{k=1}^{15}(4 + 3k)$.**

Solution:

The sum of the first n terms in an arithmetic series is given by: $S_n = n\left(\frac{a_1 + a_n}{2}\right)$

$$\begin{aligned} a_1 &= 4 + 3(1) = 7 && \textbf{Identify first term.} \\ a_{15} &= 4 + 3(15) = 49 && \textbf{Identify last term.} \\ S_{15} &= 15\left(\frac{7 + 49}{2}\right) && \textbf{Write rule for } S_{15}. \\ S_{15} &= 420 && \textbf{Simplify.} \end{aligned}$$

PRACTICE **Write a rule for the nth term of the sequence. Then find a_{18}.**

7. 1, 6, 11, 16, … **8.** 12, 21, 30, 39, … **9.** 19, 15, 11, 7, …

Find the sum of the arithmetic series.

10. $\sum_{i=1}^{12}(7 + 2i)$ **11.** $\sum_{n=1}^{30}(10 - 3n)$ **12.** $\sum_{k=4}^{25}(-1 + k)$

Name ____________________ Date ____________

BENCHMARK 5

(Chapters 10, 11 and 12)

4. Analyze Geometric Sequences and Series

Vocabulary

Geometric sequence A sequence where the ratio of any term to the previous term is constant.

Common ratio The constant ratio in a geometric sequence, denoted by r.

EXAMPLE **Write a rule for the *n*th term of the sequence. Then find a_9.**

a. 3, 12, 48, 192, … **b.** −256, 128, −64, 32, …

The rule for the nth term, a_n, in a geometric sequence is given by: $a_n = a_1 \cdot r^{n-1}$

Solution:

a. The sequence is geometric with first term $a_1 = 3$ and common ratio $r = \frac{12}{3} = 4$. So the rule for the nth term is:

$a_n = a_1 r^{n-1}$ **Write general rule.**

$= 3(4)^{n-1}$ **Substitute 3 for a_1 and 4 for r.**

The 9th term is $a_9 = 3(4)^{9-1} = 196{,}608$.

b. The sequence is geometric with first term $a_1 = -256$ and common ratio $r = -\frac{128}{256} = -\frac{1}{2}$. So the rule for the nth term is:

$a_n = a_1 r^{n-1}$ **Write general rule.**

$= -256\left(-\frac{1}{2}\right)^{n-1}$ **Substitute −256 for a_1 and $-\frac{1}{2}$ for r.**

The 9th term is $a_9 = -256\left(-\frac{1}{2}\right)^{9-1} = -1$.

EXAMPLE **Find the sum of the geometric series $\sum_{i=1}^{18} 3(2)^{i-1}$.**

The sum of the first n terms in a finite geometric series with $r \neq 1$ is given by: $S_n = a_1\left(\frac{1 - r^n}{1 - r}\right)$

Solution:

$a_1 = 3(2)^{1-1} = 3$ **Identify first term.**

$r = 2$ **Identify common ratio.**

$S_{18} = 3\left(\frac{1 - 2^{18}}{1-2}\right)$ **Write rule for S_{18}.**

$S_{18} = 786{,}429$ **Simplify.**

PRACTICE

Write a rule for the *n*th term of the sequence. Then find a_8.

13. 1, 2, 4, 8, … **14.** 2, −10, 50, −250, … **15.** 243, 81, 27, 9, …

Find the sum of the geometric series.

16. $\sum_{k=1}^{10} 6(2)^{k-1}$ **17.** $\sum_{i=1}^{9} 16\left(-\frac{1}{2}\right)^{i-1}$ **18.** $\sum_{m=0}^{7} -3^m$

Name ______________________ Date ______________

BENCHMARK 5

(Chapters 10, 11 and 12)

5. Find the Sums of Infinite Geometric Series

Vocabulary **Partial sum** The sum S_n of the first n terms in an infinite geometric series.

EXAMPLE **Find the sum of the infinite geometric series.**

The sum of an infinite geometric series with $|r| < 1$ is: $S = \frac{a_1}{1-r}$

If $|r| \geq 1$, the series has no sum.

a. $\sum_{k=1}^{\infty} 3(-0.2)^{k-1}$ **b.** $8 + 4 + 2 + 1 + \ldots$ **c.** $\frac{1}{2} - \frac{3}{4} + \frac{9}{8} - \frac{27}{16} + \ldots$

Solution:

a. $a_1 = 3$ and $r = -0.2$. So, $S = \frac{a_1}{1-r} = \frac{3}{1-(-0.2)} = 2.5$.

b. $a_1 = 8$ and $r = 0.5$. So, $S = \frac{a_1}{1-r} = \frac{8}{1-(0.5)} = 16$.

c. $a_1 = \frac{1}{2}$ and $r = \frac{-\frac{3}{4}}{\frac{1}{2}} = -\frac{3}{2}$. Because $\left|-\frac{3}{2}\right| \geq 1$, the sum does not exist.

PRACTICE **Find the sum of the infinite geometric series, if it exists.**

19. $\sum_{n=1}^{\infty} 7\left(-\frac{2}{3}\right)^{n-1}$ **20.** $\sum_{k=1}^{\infty} 2\left(\frac{4}{3}\right)^{k-1}$

21. $-1 + \frac{3}{5} - \frac{9}{25} + \frac{27}{125} - \ldots$

6. Write a Recursive Rule

Vocabulary **Recursive rule** Defines a sequence by giving the beginning term or terms of a sequence followed by a *recursive equation* that tells how a_n is related to one or more preceding terms.

EXAMPLE **Write a recursive rule for the sequence.**

The recursive equation for an arithmetic sequence is: $a_n = a_{n-1} + d$

The recursive equation for a geometric sequence is: $a_n = r \cdot a_{n-1}$

a. 4, 13, 22, 31, … **b.** 16, 24, 36, 54, … **c.** 2, 2, 4, 12, 48, …

Solution:

a. The sequence is arithmetic with first term $a_1 = 4$ and common difference $d = 13 - 4 = 9$.

$a_n = a_{n-1} + d$ **General recursive equation for a_n**

$= a_{n-1} + 9$ **Substitute 9 for d.**

So, a recursive rule for the sequence is $a_1 = 4$, $a_n = a_{n-1} + 9$.

b. The sequence is geometric with first term $a_1 = 16$ and common ratio $r = \frac{24}{16} = \frac{3}{2}$.

$a_n = r \cdot a_{n-1}$ **General recursive equation for a_n**

$= \frac{3}{2} \cdot a_{n-1}$ **Substitute $\frac{3}{2}$ for r.**

So, a recursive rule for the sequence is $a_1 = 16$, $a_n = \frac{3}{2} \cdot a_{n-1}$.

BENCHMARK 5

(Chapters 10, 11 and 12)

c. The first term is $a_0 = 2$. Note that $a_1 = 2 = 1 \cdot a_0$, $a_2 = 4 = 2 \cdot a_1$, $a_3 = 12 = 3 \cdot a_2$, and so on.

So, a recursive rule for the sequence is $a_0 = 2$, $a_n = n \cdot a_{n-1}$.

PRACTICE **Write a recursive rule for the sequence.**

22. $5, -10, 20, -40, \ldots$

23. $1, 2, 8, 48, 384, \ldots$

7. Iterate a Function

Vocabulary **Iteration** The repeated composition of a function f with itself.

EXAMPLE **Find the first three iterates of the function $f(x) = 2x - 5$ for an initial value $x_0 = 3$.**

To generate a recursive function using iteration, begin with an initial value x_0. Let $x_1 = f(x_0)$, $x_2 = f(x_1) = f(f(x_0))$, and so on.

Solution:

$$x_1 = f(x_0) = f(3) = 2(3) - 5 = 1$$

$$x_2 = f(x_1) = f(1) = 2(1) - 5 = -3$$

$$x_3 = f(x_2) = f(-3) = 2(-3) - 5 = -11$$

The first three iterates are 1, -3, and -11.

PRACTICE **Find the first three iterates of the function for the given initial value.**

24. $f(x) = -x + 4, x_0 = 5$

25. $f(x) = x^2 - 3x, x_0 = -1$

Quiz

Write the first six terms of the sequence.

1. $a_n = n - 5$

2. $f(n) = n^2 - 4$

3. $a_n = \frac{2}{n}$

Find the sum of the series.

4. $\sum_{i=1}^{8} i$

5. $\sum_{k=2}^{5} k^2 + 1$

6. $\sum_{k=4}^{7} \frac{k}{k-3}$

7. $\sum_{i=1}^{25} -5 + 2i$

8. $\sum_{j=1}^{8} 4(5)^{j-1}$

9. $\sum_{i=0}^{6} 5\left(\frac{3}{2}\right)^i$

Write a rule for the *n*th term of the sequence. Then find a_{10}.

10. $1, 9, 17, 25, \ldots$

11. $103, 96, 89, 82, \ldots$

12. $3, 6, 12, 24, \ldots$

13. $-1, 4, -16, 64, \ldots$

Name ______________________ Date ______________

BENCHMARK 5
(Chapters 10, 11 and 12)

Find the sum of the infinite geometric series, if it exists.

14. $\sum_{n=1}^{\infty} 3\left(\frac{3}{8}\right)^{n-1}$

15. $18 - 12 + 8 - \frac{16}{3} + \ldots$

Write a recursive rule for the sequence.

16. $25, 20, 15, 10, \ldots$

17. $4, 10, 25, 62.5, \ldots$

Find the first three iterates of the function for the given initial value.

18. $f(x) = -\frac{1}{2}x - 2, x_0 = 8$

19. $f(x) = x^2 + 3, x_0 = 1$

BENCHMARK 6
(Chapters 13 and 14)

A. Trigonometric Functions (pp. 118–123)

Ratios of the sides of a right triangle are used to define the six trigonometric functions. These trigonometric functions, in turn, are used to help find unknown side lengths and angle measures of triangles.

1. Evaluate Trigonometric Functions

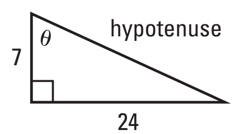
Vocabulary
The six trigonometric functions and their abbreviations are:
sine, or sin
cosine, or cos
tangent, or tan
cotangent, or cot
secant, or sec
cosecant, or csc

In a right triangle with acute angle θ, the six trigonometric functions are defined as follows:

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite side}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent side}} \qquad \cot\theta = \frac{\text{adjacent side}}{\text{opposite side}}$$

EXAMPLE **Evaluate the six trigonometric functions of the angle θ.**



Solution:

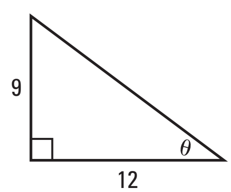
From the Pythagorean Theorem, the hypotenuse has length $\sqrt{7^2 + 24^2} = \sqrt{625} = 25$.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{24}{25} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{7}{25} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{24}{7}$$

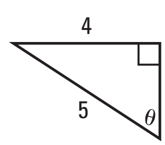
$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{25}{24} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{25}{7} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{7}{24}$$

PRACTICE **Evaluate the six trigonometric functions of the angle θ.**

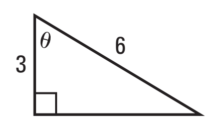
1.



2.



3.

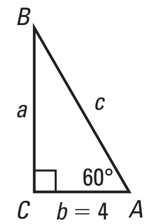


Name ______________________________ Date ______________

BENCHMARK 6
(Chapters 13 and 14)

2. Solve a Right Triangle

EXAMPLE **Solve $\triangle ABC$.**



Solution:

A and B are complementary angles, so $B = 90° - 60° = 30°$.

$\csc\theta = \frac{1}{\sin\theta}$

$\sec\theta = \frac{1}{\cos\theta}$

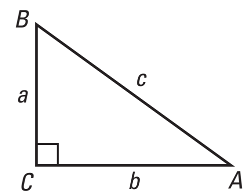
$\cot\theta = \frac{1}{\tan\theta}$

$\tan 60° = \frac{\text{opp}}{\text{adj}}$	$\sec 60° = \frac{\text{hyp}}{\text{adj}}$	**Write trigonometric equation.**
$\tan 60° = \frac{a}{4}$	$\sec 60° = \frac{c}{4}$	**Substitute.**
$a = 4(\tan 60°)$	$c = 4(\sec 60°) = 4\left(\frac{1}{\cos 60°}\right)$	**Solve for the variable.**
$a = 4\sqrt{3}$	$c = 4\left(\frac{1}{\frac{1}{2}}\right) = 4 \cdot 2 = 8$	**Evaluate trigonometric functions.**

PRACTICE **Solve $\triangle ABC$ using the diagram and given measurements.**

4. $B = 45°, c = 12\sqrt{2}$

5. $A = 30°, b = 6$



3. Draw General Angles

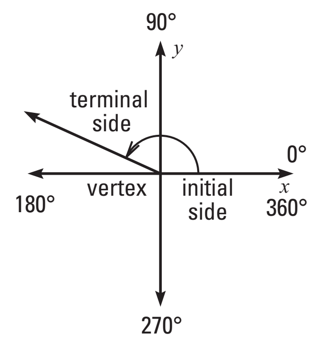
Vocabulary

The measure of an angle is positive if the rotation of the terminal side is counterclockwise. If the rotation is clockwise, the angle measure is negative.

Initial side The fixed ray of an angle on a coordinate plane.

Terminal side The ray of an angle that is rotated about the vertex on a coordinate plane.

Standard position The location of an angle whose vertex is at the origin and initial side lies on the positive x-axis.

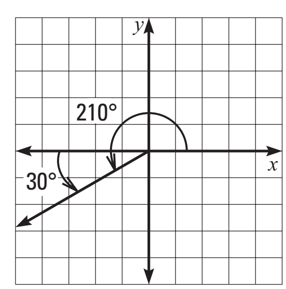


EXAMPLE **Draw an angle with the given measure in standard position.**

a. 210° **b.** 480° **c.** $-135°$

Solution:

a. Since 210° is 30° more than 180°, the terminal side is 30° counterclockwise past the negative x-axis.

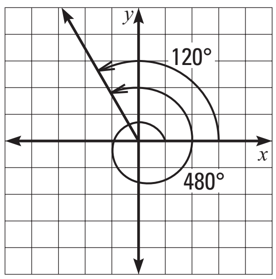


Name ______________________ Date ______________

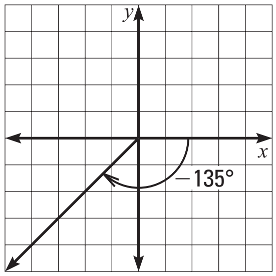
BENCHMARK 6

(Chapters 13 and 14)

b. Since 480° is 120° more than 360°, the terminal side makes one whole revolution counterclockwise plus 120° more.



c. Since –135° is negative, the terminal side is 135° clockwise from the positive x-axis.



PRACTICE **Draw an angle with the given measure in standard position.**

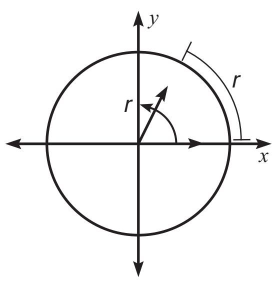
6. 300° **7.** −225° **8.** 600°

4. Convert Between Degrees and Radians

Vocabulary **Radian** The measure of an angle in standard position whose terminal side intercepts an arc of length r.

To convert degrees to radians, multiply degrees by $\frac{\pi \text{ radians}}{180°}$.

To convert radians to degrees, multiply radians by $\frac{180°}{\pi \text{ radians}}$.



EXAMPLE **Convert (a) 225° to radians and (b) $\frac{2\pi}{3}$ to degrees.**

Solution:

a. $225° = 225° \cdot \frac{\pi \text{ radians}}{180°} = \frac{5\pi}{4}$ radians, or simply $\frac{5\pi}{4}$

b. $\frac{2\pi}{3} = \frac{2\pi}{3} \text{ radians} \cdot \frac{180°}{\pi \text{ radians}} = 120°$

PRACTICE **Convert the degree measure to radians or the radian measure to degrees.**

9. 150° **10.** $\frac{7\pi}{6}$ **11.** $\frac{\pi}{9}$

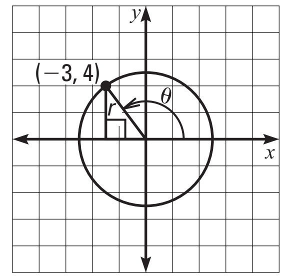
BENCHMARK 6
A. Trigonometric Functions

Name ______________________________ Date ______________

BENCHMARK 6
(Chapters 13 and 14)

5. Evaluate Trigonometric Functions Given a Point

EXAMPLE **The point $(-3, 4)$ is a point on the terminal side of an angle θ in standard position. Evaluate the six trigonometric functions of θ.**



For an angle θ in standard position whose terminal side intersects a circle with radius r, the trigonometric functions are defined as follows:
$\sin\theta = \frac{y}{r}$
$\cos\theta = \frac{x}{r}$
$\tan\theta = \frac{y}{x}, x \neq 0$

Solution:

By the Pythagorean Theorem,

$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5.$

For $x = -3$, $y = 4$, and $r = 5$,

$\sin\theta = \frac{y}{r} = \frac{4}{5}$ $\quad \cos\theta = \frac{x}{r} = -\frac{3}{5}$ $\quad \tan\theta = \frac{y}{x} = -\frac{4}{3}$

$\csc\theta = \frac{1}{\sin\theta} = \frac{5}{4}$ $\quad \sec\theta = \frac{1}{\cos\theta} = -\frac{5}{3}$ $\quad \cot\theta = \frac{1}{\tan\theta} = -\frac{3}{4}$

PRACTICE **Evaluate the six trigonometric functions with the given point on the terminal side of angle θ.**

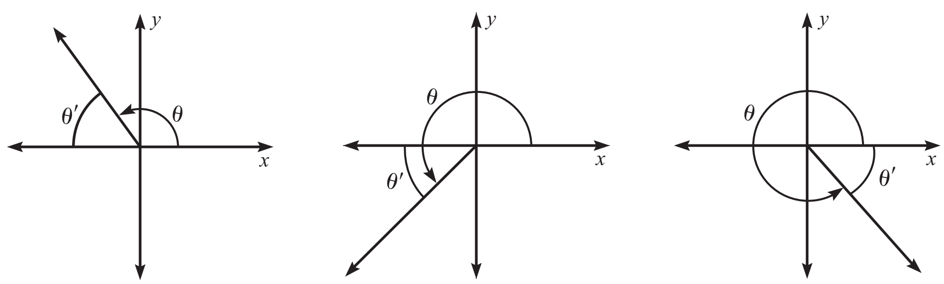
12. $(6, 8)$ **13.** $(8, -15)$ **14.** $(-12, -5)$

6. Use Reference Angles

Vocabulary **Reference angle** The acute angle θ' formed by the terminal side of an angle θ in the standard position and the x-axis.

Signs of trigonometric functions in each quadrant are as follows:

Quadrant I
$\sin\theta = +$
$\cos\theta = +$
$\tan\theta = +$



EXAMPLE **Use reference angles to evaluate (a) $\tan(-225°)$ and (b) $\cos\frac{8\pi}{3}$.**

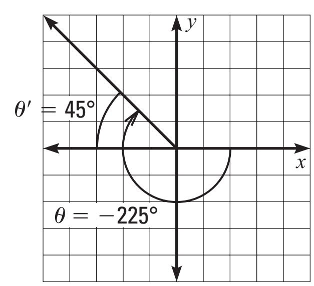
Quadrant II
$\sin\theta = +$
$\cos\theta = -$
$\tan\theta = -$

Quadrant III
$\sin\theta = -$
$\cos\theta = -$
$\tan\theta = +$

Quadrant IV
$\sin\theta = -$
$\cos\theta = +$
$\tan\theta = -$

Solution:

a. The angle $-225°$ is coterminal with $135°$.
The reference angle is $180° - 135° = 45°$.
The tangent function is negative in Quadrant II,
so $\tan(-225°) = -\tan 45° = -1$.



Name ______________________ Date ______________

BENCHMARK 6

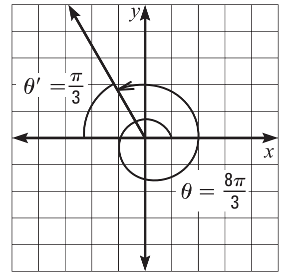
(Chapters 13 and 14)

b. The angle $\frac{8\pi}{3}$ is coterminal with $\frac{2\pi}{3}$.

The reference angle is $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$.

The cosine function is negative in Quadrant II,

so $\cos \frac{8\pi}{3} = -\cos \frac{\pi}{3} = -\frac{1}{2}$.



PRACTICE **Evaluate the trigonometric function without using a calculator.**

15. $\cos 300°$ **16.** $\csc -330°$ **17.** $\tan \frac{17\pi}{6}$

7. Evaluate Inverse Trigonometric Functions, Solve a Trigonometric Equation

EXAMPLE **Evaluate (a) $\sin^{-1} 0.5$ and (b) $\tan^{-1} \sqrt{3}$.**

For $\sin \theta = a$, the **inverse sine** is $\sin^{-1} a = \theta$, $-90° \le \theta \le 90°$.

For $\cos \theta = a$, the **inverse cosine** is $\cos^{-1} a = \theta$, $0° \le \theta \le 180°$.

For $\tan \theta = a$, the **inverse tangent** is $\tan^{-1} a = \theta$, $-90° < \theta < 90°$.

Solution:

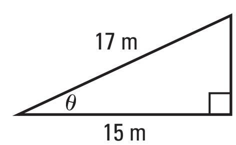
a. When $-90° \le \theta \le 90°$, the angle whose sine is 0.5 is:
$\theta = \sin^{-1} 0.5 = 30°$

b. When $-90° < \theta < 90°$, the angle whose tangent is $\sqrt{3}$ is:
$\theta = \tan^{-1} \sqrt{3} = 60°$

EXAMPLE **A 17-meter ramp has a horizontal length of 15 meters. What is the angle θ of the ramp?**

Solution:

Draw a triangle that represents the ramp. Write a trigonometric equation that involves the ramp's length and horizontal length. $\cos \theta = \frac{15}{17}$



Use a calculator to find the measure of θ.

$$\theta = \cos^{-1} \frac{15}{17} \approx 28.1°$$

PRACTICE **Evaluate the expression.**

18. $\tan^{-1} -1$ **19.** $\cos^{-1} 0.5$ **20.** $\sin^{-1} \sqrt{2}$

21. A cable wire is attached to the top of a 10-foot pole 6 feet from the base of the pole. What is the angle the wire makes with the ground?

Name ____________________ Date ____________

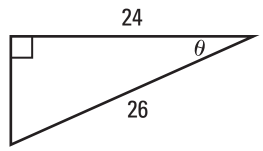
BENCHMARK 6

(Chapters 13 and 14)

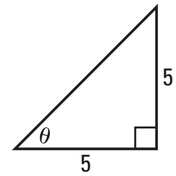
Quiz

Evaluate the six trigonometric functions of the angle θ.

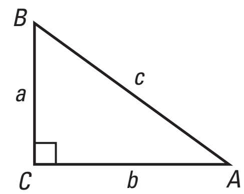
1.



2.



Solve $\triangle ABC$ using the diagram and given measurements.



3. $A = 45°, a = 11$

4. $B = 60°, c = 15$

Convert the degree measure to radians or the radian measure to degrees.

5. $75°$

6. $\frac{11\pi}{3}$

7. $250°$

Evaluate the six trigonometric functions with the given point on the terminal side of angle θ.

8. $(-20, 15)$

9. $(-24, -7)$

10. $(2, -2)$

Evaluate without using a calculator.

11. $\tan(-120°)$

12. $\cos \frac{5\pi}{2}$

13. $\cos^{-1} \frac{\sqrt{2}}{2}$

14. $\sin^{-1} -\frac{\sqrt{3}}{2}$

15. An airplane begins its descent for landing at an altitude of 29,000 feet. At this time, the airplane is 50 miles from the runway. At what angle does the airplane descend?