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BENCHMARK 3
(Chapters 5 and 6)

A. Polynomials

Polynomials are one of the most fundamental types of functions used in mathematics. They are very simple to use, primarily because they are formed entirely by multiplication (exponents are repeated multiplications) and addition. Becoming familiar with basic operations using polynomials is critical for later work.

1. Use Properties of Exponents

Vocabulary

Scientific notation A number is expressed in scientific notation if it is in the form $c \times 10^n$ where $1 \le c < 10$ and n is an integer.

EXAMPLE **Use the properties of exponents.**

a. Evaluate $\left(\frac{5^2 \cdot 5^3}{5}\right)^3$

b. Water weighs approximately 0.00205 pounds per cubic centimeter. What is the weight of the water in a small swimming pool containing 28,000,000 cubic centimeters of water?

c. Simplify $\left(\frac{3a^2b}{15b^{-3}}\right)^{-2}$

Solution:

a. $\left(\frac{5^2 \cdot 5^3}{5}\right)^3 = \left(\frac{5^5}{5}\right)^3$ **Product of power property**

$= (5^{5-1})^3$ **Quotient of powers property**

$= (5^4)^3 = 5^{12}$ **Power of a power property**

Be sure your final answer is in scientific notation. That is, make sure the first number is between 1 included and 10 not included.

b. Weight of water $=$ Weight per $\text{cm}^3 \times$ Number of cm^3

$= 0.00205 \times 28{,}000{,}000$ **Substitute values.**

$= (2.205 \times 10^{-3})(2.8 \times 10^7)$ **Write in scientific notation.**

$= (2.205 \times 2.8)(10^{-3} \times 10^7)$ **Use multiplication properties**

$= 6.174 \times 10^4$ **Product of power property**

The weight of the water in the pool is about 61,740 pounds.

There are many different ways to arrive at the correct answer. With practice, you will begin to develop a personal style. Just make sure to justify each step with a property.

c. $\left(\frac{3a^2b}{15b^{-3}}\right)^{-2} = \left(\frac{a^2b}{5b^{-3}}\right)^{-2}$ **Simplify quantity.**

$= \left(\frac{a^2b^{1-(-3)}}{5}\right)^{-2}$ **Quotient of powers property**

$= \left(\frac{5}{a^2b^4}\right)^2$ **Negative exponent property**

$= \frac{5^2}{(a^2b^4)^2}$ **Power of a quotient property**

$= \frac{25}{a^4b^8}$ **Power of a product and power of a power property**

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BENCHMARK 3

(Chapters 5 and 6)

PRACTICE **Evaluate the expression.**

1. $(3^2)^2(-2)$ **2.** $\left(\frac{8^{-2}}{8^{-3}}\right)^{-2}$ **3.** $(2^0 \cdot 3 \cdot 2^{-2})^3$

Simplify the expression.

4. $(a^2b^{-2}c^{-5})^{-3}$ **5.** $\left(\frac{x^2y^{-5}}{x^{-3}}\right)^{-2}$ **6.** $\frac{3r^2s^5}{12r^0s^{-1}}$

7. The weight of the water required to fill a cylindrical tank is 100 pounds.Use the fact that water weighs approximately 0.00205 pounds per cubic centimeter to calculate the volume of the tank.

2. Identify Polynomial Functions

Vocabulary

Monomial A number, a variable, or a product of numbers and variables.

Coefficient of a monomial The number in the monomial, and it is usually written in front of any variables.

Polynomial A monomial or a sum of monomials.

Polynomial function A function that can be written in the form $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ where $a_n \neq 0$, the exponents are all whole numbers, and the coefficients are all real numbers.

Standard form of a polynomial function A form of a polynomial function where the terms are written in descending order of exponents from left to right.

EXAMPLE **Decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.**

a. $f(x) = 1 - x$ **b.** $g(x) = x^{3/2} + 2x^{1/2} - 5$

c. $h(x) = 5x^3 - 2x^2 + x$ **d.** $k(x) = \pi$

Solution:

The leading coefficient of −1 in example a is hidden. Anytime there is no apparent coefficient, consider what it means to multiply by 1.

a. The function is a polynomial function written as $f(x) = -x + 1$ in standard form. It has degree 1 (linear) and a leading coefficient of −1.

b. The function is not a polynomial function because the exponents of x are not whole numbers.

c. The function is a polynomial function already written in standard form. It has degree 3 (cubic) and a leading coefficient of 5.

d. The function is a polynomial function already written in standard form. It has degree 0 (constant) and a leading coefficient of π.

PRACTICE **Decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.**

8. $f(x) = 5 - 3x + 4x^2$ **9.** $g(x) = 5x^{-4} + 2x^3 - 4$ **10.** $h(x) = x^3 + 3^x$

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BENCHMARK 3

(Chapters 5 and 6)

3. Evaluate Polynomials

EXAMPLE **Evaluate $f(x) = x^5 - 2x^2 + 3x - 11$ when $x = -3$ using (a) direct substitution and (b) synthetic substitution.**

Solution:

Be careful to correctly evaluate the polynomial function when *x* is a negative value.

a. $f(x) = x^5 - 2x^2 + 3x - 11$ — **Write original function.**

$f(-3) = (-3)^5 - 2(-3)^2 + 3(-3) - 11$ — **Substitute −3 for *x*.**

$= -243 - 18 - 9 - 11$ — **Evaluate powers and multiply.**

$= -281$ — **Simplify.**

Notice that synthetic substitution requires no exponent calculations.

b. Write the coefficients of $f(x)$ in order of descending exponents. Write the value at which $f(x)$ is being evaluated to the left. Follow the steps outlined on page 338 of the textbook.

−3	1	0	0	−2	3	−11
		−3	9	−27	87	−270
	1	−3	9	−29	90	**−281**

The final number on the last line is the answer, −281. This agrees with our result in part (a).

PRACTICE **Evaluate the function for the given value of *x*, using both direct substitution and synthetic substitution.**

11. $f(x) = -2x^3 + x^2 - 5x + 6; x = -2$ **12.** $g(x) = 5x^4 - 2; x = 4$

4. Add, Subtract, and Multiply Polynomials

EXAMPLE **(a) Add, (b) subtract, and (c) multiply $3x^3 - 2x + 1$ and $-x^2 + 5x - 2$.**

Solution:

In part b, remember to switch from subtraction to addition by changing *all* the signs in the second polynomial.

a.

$$\begin{array}{r} 3x^3 \qquad\quad - 2x + 1 \\ + \quad -x^2 + 5x - 2 \\ \hline 3x^3 - x^2 + 3x - 1 \end{array}$$

b.

$$\begin{array}{r} 3x^3 \qquad\quad -2x + 1 \\ - \quad (-x^2 + 5x - 2) \\ \hline \end{array} \Rightarrow \begin{array}{r} 3x^3 \qquad - 2x + 1 \\ + \quad x^2 - 5x + 2 \\ \hline 3x^3 + x^2 - 7x + 3 \end{array}$$

c.

$$\begin{array}{r} 3x^3 \qquad\quad - 2x + 1 \\ \times \qquad -x^2 + 5x - 2 \\ \hline -6x^3 \qquad\quad + 4x - 2 \\ 15x^4 \qquad -10x^2 + 5x \qquad \\ -3x^5 \qquad + 2x^3 - x^2 \qquad\qquad \\ \hline -3x^5 + 15x^4 - 4x^3 - 11x^2 + 9x - 2 \end{array}$$

Multiply $3x^3 - 2x + 1$ by −2.

Multiply $3x^3 - 2x + 1$ by $5x$.

Multiply $3x^3 - 2x + 1$ by $-x^2$.

Combine like terms.

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BENCHMARK 3

(Chapters 5 and 6)

PRACTICE **Perform the operation.**

13. $(5x^2 - 2) + (-3x^2 - 2x)$

14. $(-6x^3 - 2x^2 + x) - (3x^3 - x + 5)$

15. $(3x - 2)(x^3 - 2x^2 + x - 3)$

16. $(2x^2 - 3x + 1)(-x^2 + 5x - 2)$

5. Use Special Product Patterns

Vocabulary **Special Product Patterns**

1. Sum and Difference $(a + b)(a - b) = a^2 - b^2$

2. Square of a Binomial $(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

3. Cube of a Binomial $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

EXAMPLE **Use special product patterns to find the product.**

a. $(2x - 5)(2x + 5)$ **b.** $(3x + 2y)^2$ **c.** $(2x - 3)^3$

Solution:

Remember multiplication is commutative. To use the sum and difference pattern in part a, we do not need $(2x + 5)$ first.

a. $(2x - 5)(2x + 5) = (2x)^2 - (5)^2$ **Sum and difference.**

$= 4x^2 - 25$

b. $(3x + 2y)^2 = (3x)^2 + 2(3x)(2y) + (2y)^2$ **Square of a binomial.**

$= 9x^2 + 12xy + 4y^2$

c. $(2x - 3)^3 = (2x)^3 - 3(2x)^2(3) + 3(2x)(3)^2 + (3)^3$ **Cube of a binomial.**

$= 8x^3 - 36x^2 + 54x + 27$

PRACTICE **Use special product patterns to find the product.**

17. $(x - 9)^2$ **18.** $(7a + 3b)^2$ **19.** $(4x + 1)^3$ **20.** $(4t - 1)(4t + 1)$

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BENCHMARK 3
(Chapters 5 and 6)

6. Use Polynomial Long Division

EXAMPLE **Divide $f(x) = 19x^2 - 7x^3 + 10x^4 - 6x + 11$ by $2x^2 - x + 2$**

Solution:

Before beginning, make sure that both of the polynomials are written in descending powers. Also, be sure to include 0 as the coefficient of any terms that may be missing.

When performing the subtractions, be sure to subtract each term. A common error is to subtract only the leading terms.

$$\begin{array}{r} 5x^2 - x + 4 \\ 2x^2 - x + 2 \overline{)10x^4 - 7x^3 + 19x^2 - 6x + 11} \\ \underline{10x^4 - 5x^3 + 10x^2} \\ -2x^3 + 9x^2 - 6x + 11 \\ \underline{-2x^3 + x^2 - 2x} \\ 8x^2 - 4x + 11 \\ \underline{8x^2 - 4x + 8} \\ 3 \end{array}$$

← **Quotient.**

Multiply divisor by $\frac{10x^4}{2x^2} = 5x^2$.

Subtract.

Multiply divisor by $-\frac{2x^3}{2x^2} = -x$.

Subtract.

Multiply divisor by $\frac{8x^2}{2x^2} = 4$.

← **Remainder.**

$$\frac{10x^4 - 7x^3 + 19x^2 - 6x + 11}{2x^2 - x + 2} = 5x^2 - x + 4 + \frac{3}{2x^2 - x + 2}$$

PRACTICE **Use polynomial long to divide.**

21. $(2x^4 - 12x^2 + 12) \div (2x^2 - 3)$

22. $(9x^4 + 6x^3 - 2x^2 - 8x - 8) \div (3x^2 + 2x + 1)$

23. $\dfrac{6x^4 - x^3 - 18x^2 + 5x - 1}{2x^2 - 6}$

Quiz

1. Evaluate $\dfrac{(3^5 \cdot 2^3)^2}{4 \cdot 3^7}$

2. Simplify $\dfrac{(x^{-2}y^4)^3}{x^{-4}y^7}$

3. A company spends 2.4 million dollars to manufacture 640,000 units of a particular brand of toy. How much does it cost to manufacture one toy?

Decide whether the function is a polynomial function. If so, write it in standard form and state its degree, type, and leading coefficient.

4. $f(x) = x^{-5} + 3x^2 + 1$ **5.** $h(x) = \sqrt{3}x^2 - \frac{x^3}{4} - x$ **6.** $g(x) = 3^x + 2x - 5$

7. Evaluate $f(x) = -x^3 + 4x^2 - 2x + 10$ when $x = 4$.

8. Add $3x^3 - 2x^2 + 5$ and $x^3 - 3x + 5$.

9. Subtract $2z^2 - 4z - 10$ from $5z^2 - 3z + 25$.

10. Multiply and $3y^2 - 2y + 5$ and $3y + 1$.

11. Find the product $(a - 3b)^3$.

12. Divide $x^4 + 2x^3 - 4x^2 + x - 12$ by $x^2 - 2$.

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BENCHMARK 3

(Chapters 5 and 6)

B. Graphing and Writing Polynomial Functions

Every function has a graph. Some are more complicated than others, but polynomial functions are among the most simple. There are various relationships between the algebraic expression of a polynomial and its graph. Several of these relationships are summarized here.

1. Graph Polynomial Functions

Vocabulary

End behavior The behavior of a function's graph as x approaches positive infinity ($+\infty$) or negative infinity ($-\infty$). It is determined by the degree and leading coefficient of the polynomial.

EXAMPLE **Graph the polynomial function.**

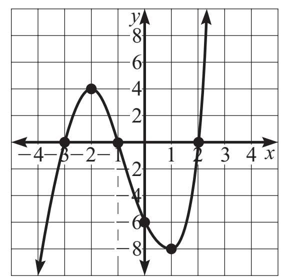
a. $f(x) = x^3 + 2x^2 - 5x - 6$ **b.** $g(x) = -\frac{1}{4}(x + 2)(x)(x - 3)^2$

Solution:

When the function is given in factored form, as in example b, we can quickly determine the x-intercepts by looking at the factors.

When it is written in standard form as in example a, we can quickly determine the y-intercept by looking at the constant term.

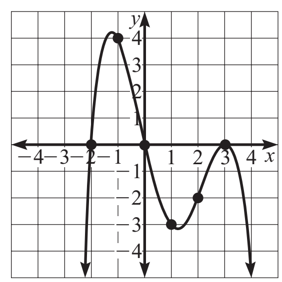
a. To graph the function, make a table of values and plot the corresponding points. Connect the points with a smooth curve and check the end behavior.



x	−3	−2	−1	0	1	2	3
y	0	4	0	−6	−8	0	24

The degree is odd and leading coefficient is positive. So, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

b. Begin by plotting the x-intercepts. Because −2, 0, and 3 are zeros of f, plot (−2, 0), (0, 0) and (3, 0). Then make a table of values between and beyond the x-intercepts, and plot these points.



x	−3	−1	1	2	4
y	−27	4	−3	−2	−6

The degree is even and leading coefficient is negative. So, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$.

PRACTICE **Graph the polynomial function.**

1. $f(x) = -15x^3 - 16x^2 + x + 2$ **2.** $g(x) = 2x^2 + 4x - 30$

3. $h(x) = 5x^2(x + 4)(x - 3)$ **4.** $j(x) = 0.5(x - 2)^2(x - 5)$

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BENCHMARK 3

(Chapters 5 and 6)

2. Identify Turning Points

Vocabulary

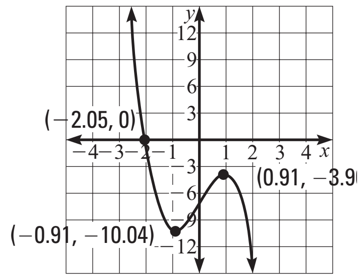
Turning point Point on the graph of a polynomial function that corresponds to a local maximum or minimum value. The graph of every polynomial function of degree n has at most $n - 1$ turning points.

EXAMPLE **Graph the function $f(x) = -2x^3 + 5x - 7$. Identify the x-intercepts and the points where the local maximums and local minimums occur.**

Be prepared to use the graphing calculator's zoom feature if the graph has a turning point near the x-axis. There may be 0, 1, or 2 zeros near that point.

Solution:

Use a graphing calculator to graph the function. Notice that the graph of f has one x-intercept and two turning points. You can use the graphing calculator's zero, maximum, and minimum features to approximate the coordinates of the points.



The x-intercept of the graph is $x \approx -2.05$. The function has a local maximum at $(0.91, -3.96)$ and a local minimum at $(-0.91, -10.04)$.

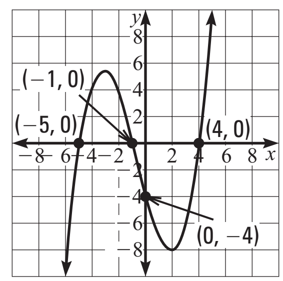
PRACTICE **Graph the following functions, and identify the x-intercepts and the points where the local maximums and local minimums occur.**

5. $f(x) = x^4 - 3x^2 + 10x - 5$

6. $g(x) = -3x^3 + 10x$

3. Write a Cubic Function

EXAMPLE **Write the cubic function whose graph is shown.**



Solution:

If a cubic function has a turning point at a zero, write the corresponding factor twice when writing the function in factored form.

STEP 1: Use the three given x-intercepts to write the function in factored form.

$f(x) = a(x + 5)(x + 1)(x - 4)$

STEP 2: Find the value of a by substituting the coordinate of the fourth point.

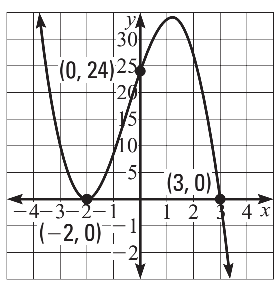
$-4 = a(0 + 5)(0 + 1)(0 - 4)$

$-4 = -20a$

$\frac{1}{5} = a$

The function is $f(x) = \frac{1}{5}(x + 5)(x + 1)(x - 4)$.

PRACTICE **7.** Write the cubic function whose graph is shown.



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BENCHMARK 3

(Chapters 5 and 6)

4. Find Finite Differences

Vocabulary

Finite differences The differences of consecutive y-values when the x-values in a data set are equally spaced.

EXAMPLE

Show that the third-order differences for the degree 3 function $f(x) = x^3 + 5x - 1$ are nonzero and constant.

When choosing x-values, consider the degree of the function. In this example, evaluating differences like $-19 - (-43)$ may be preferable to evaluating $f(7)$.

Write the first few function values. Find the first-order differences by subtracting consecutive function values. Then find the second-order differences by subtracting consecutive first-order differences. Finally, find the third-order differences by subtracting consecutive second-order differences.

	$f(-3)$	$f(-2)$	$f(-1)$	$f(0)$	$f(1)$	$f(2)$	$f(3)$
Write function values	−43	−19	−7	−1	5	17	41
1st-order differences	24	12	6	6	12	24	
2nd-order differences	−12	−6	0	6	12		
3rd-order differences	6	6	6	6			

Each third-order difference is 6, so the third-order differences are constant.

PRACTICE

Show that the *n*th-order differences for the given function of degree *n* are nonzero and constant.

8. $f(x) = -6x + 8$

9. $g(x) = 3x^2 - x + 5$

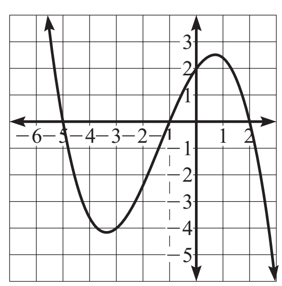
10. $h(x) = x^3 - 2x^2 - 3$

11. $j(x) = x^4 - 16$

Quiz

1. Graph the polynomial function $f(x) = -x^3 - 4x^2 + 3x + 18$. Identify the x-intercepts and the points where the local maximums and local minimums occur.

2. Write the cubic function whose graph is shown.



3. Show that the 4th-order differences for $f(x) = x^4 - 2x^2 - 1$ are nonzero and constant.

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BENCHMARK 3
C. Factoring Polynomials

C. Factoring Polynomials

Although there are many ways to write a polynomial, the two principal forms are the standard form and the factored form. Successfully factoring a polynomial is an invaluable skill that must be mastered. Although often neglected, understanding the basic idea of what factoring is accomplishing can be critical for success. When a polynomial is factored, it is being expressed as a *product*. This is different from standard form, where the polynomial is a *sum* of terms.

1. Factor the Sum or Difference of Two Cubes; Factor by Grouping; Factor Polynomials in Quadratic Form

Vocabulary

Factored completely A factorable polynomial with integer coefficients is factored completely if it is written as a product of unfactorable polynomials with integer coefficients.

EXAMPLE **Factor the polynomial completely.**

a. $8x^4 - 27xy^6$ **b.** $2x^3 + 10x^2 - 18x - 90$ **c.** $x^6 - 2x^3 - 3$

Note that the patterns for the sum and difference of two cubes are very similar. They differ only in the signs between the terms. They become the same pattern by using the acronym S.O.A.P. This stands for S.ame sign (as original binomial) – O.pposite sign (as original binomial) – A.lways P.ositive.

Solution:

a. $8x^4 - 27xy^6 = x(8x^3 - 27y^6)$ **Factor common monomial.**

$= x[(2x)^3 - (3y^2)^3]$ **Difference of two cubes.**

$= x(2x - 3y^2)(4x^2 + 6xy^2 + 9y^4)$

b. $2x^3 + 10x^2 - 18x - 90 = 2(x^3 + 5x^2 - 9x - 45)$ **Factor common monomial.**

$= 2[x^2(x + 5) - 9(x + 5)]$ **Factor by grouping.**

$= 2(x^2 - 9)(x + 5)$ **Distributive property**

$= 2(x + 3)(x - 3)(x + 5)$ **Difference of two squares.**

c. $x^6 - 2x^3 - 3 = (x^3)^2 - 2(x^3) - 3$ **Write in quadratic form.**

$= (x^3 - 3)(x^3 + 1)$ **Factor quadratic form trinomial.**

$= (x^3 - 3)(x + 1)(x^2 - x + 1)$ **Sum of two cubes.**

PRACTICE **Factor the polynomial completely.**

1. $10x^3 + 640$ **2.** $16y^9 - 2y^3$

3. $81a^4 - b^4$ **4.** $x^2(x^2 + 5x - 14) - 25(x^2 + 5x - 14)$

2. Factor a Polynomial Using Synthetic Division

Theorem

Remainder theorem If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$.

Factor theorem A polynomial $f(x)$ has a factor of $x - k$ if and only if $f(k) = 0$.

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EXAMPLE **Factor $f(x) = 4x^3 + x^2 - 123x + 90$ completely given that $x - 5$ is a factor.**

Solution:

Because $x - 5$ is a factor of $f(x)$, you know that $f(5) = 0$. Use synthetic division to find the other factors.

5	4	1	−123	90
		20	105	−90
	4	21	−18	0

coefficients of quotient → (4, 21, −18)   0 ← remainder

When using synthetic division, the quotient is a polynomial with one degree less than the original polynomial.

Use the result to write $f(x)$ as a product of two factors and then factor completely.

$f(x) = 4x^3 + x^2 - 123x + 90$ **Write original polynomial.**

$= (x - 5)(4x^2 + 21x - 18)$ **Write as a product of two factors.**

$= (x - 5)(4x - 3)(x + 6)$ **Factor trinomial.**

PRACTICE

5. Factor $f(x) = x^3 - 6x^2 + 12x - 8$ completely given that $x - 2$ is a factor.

6. Factor $f(x) = x^3 - 27$ completely given that $x - 3$ is a factor.

7. Factor $f(x) = x^4 - 4x^3 - 11x^2 + 14x + 24$ completely given that $x + 1$ and $x - 4$ are both factors. (Hint: you may need to do synthetic division twice.)

3. List Possible Rational Zeros

Theorem

The rational zero theorem If $f(x) = a_nx^n + \cdots + a_1x + a_0$ has integer coefficients, then every rational zero of f has the following form:

$$\frac{p}{q} = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$$

EXAMPLE **List the possible rational zeros of *f* using the rational zero theorem.**

a. $f(x) = 10x^3 - 3x^2 + x - 1$ **b.** $f(x) = 9x^5 - 2x^4 + 6$

Solution:

a. Factors of the constant term: ± 1

Factors of the leading coefficient: $\pm 1, \pm 2, \pm 5, \pm 10$

Possible rational zeros: $\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{1}{5}, \pm\frac{1}{10}$

Simplified list of possible zeros: $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{5}, \pm\frac{1}{10}$

b. Factors of the constant term: $\pm 1, \pm 2, \pm 3, \pm 6$

Factors of the leading coefficient: $\pm 1, \pm 3, \pm 9$

Possible rational zeros: $\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{6}{1}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{3}{3}, \pm\frac{6}{3}, \pm\frac{1}{9}, \pm\frac{2}{9}, \pm\frac{3}{9}, \pm\frac{6}{9}$

Simplified list of possible zeros: $\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{1}{9}, \pm\frac{2}{9}$

Recall that these are not all zeros of the polynomial. This is merely a list of possible rational zeros. It is your task to determine which values, if any, are zeros.

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(Chapters 5 and 6)

PRACTICE **List the possible rational zeros of *f* using the rational zero theorem.**

8. $f(x) = 3x^4 - 2x^3 + 11x^2 - 17$

9. $f(x) = 15x^9 - 8x^6 + 5x^3 + 25$

4. Find the Zeros of a Polynomial Function

Vocabulary

Irrational conjugates If a and b are two rational numbers such that $\sqrt{b}$ is irrational, then $a + \sqrt{b}$ and $a - \sqrt{b}$ are called irrational conjugates. If f is a polynomial function with real coefficients, and $a + \sqrt{b}$ is a zero of f, then $a - \sqrt{b}$ is also a zero of f.

Complex conjugates Two complex numbers $a + bi$ and $a - bi$ are called complex conjugates. If f is a polynomial function with real coefficients, and $a + bi$ is an imaginary zero of f, then $a - bi$ is also a zero of f.

EXAMPLE **Find all zeros of the function.**

a. $f(x) = 6x^4 - 25x^3 + 7x^2 + 42x - 20$ **b.** $g(x) = 3x^3 - 10x^2 + 7x + 10$

Solution:

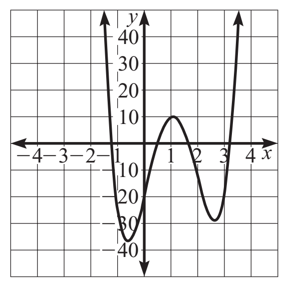
a. **STEP 1:** List the possible rational zeros of f:

$\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20, \pm\frac{1}{2}, \pm\frac{5}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{5}{3},$
$\pm\frac{10}{3}, \pm\frac{20}{3}, \pm\frac{1}{6}, \pm\frac{5}{6}$

Without a relatively accurate graph, we would need to check each possible rational zero individually!

STEP 2: Choose reasonable values from the list above to check, using the graph of the function.

For f, the values $x = \frac{1}{2}, -\frac{4}{3}$, and $\frac{5}{3}$ are reasonable based on the graph shown at the right.



STEP 3: Check the values using synthetic division until a zero is found.

$$\begin{array}{c|ccccc} \frac{1}{2} & 6 & -25 & 7 & 42 & -20 \\ & & 3 & -11 & -2 & 20 \\ \hline & 6 & -22 & -4 & 40 & 0 \end{array} \quad \longleftarrow \frac{1}{2} \text{ is a zero}$$

STEP 4: Factor out a binomial using the result of the synthetic division.

$f(x) = \left(x - \frac{1}{2}\right)(6x^3 - 22x^2 - 4x + 40)$ **Write as a product of factors.**

$= \left(x - \frac{1}{2}\right)(2)(3x^3 - 11x^2 - 2x + 20)$ **Factor 2 out of the second factor.**

$= (2x - 1)(3x^3 - 11x^2 - 2x + 20)$ **Multiply the first factor by 2.**

STEP 5: Repeat the steps above for $g(x) = 3x^3 - 11x^2 - 2x + 20$. Any zero of g will also be a zero of f. Try the remaining two reasonable rational zeros.

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BENCHMARK 3

(Chapters 5 and 6)

$$\begin{array}{r|rrrr} -\frac{4}{3} & 3 & -11 & -2 & 20 \\ & & -4 & 20 & -24 \\ \hline & 3 & -15 & 18 & -4 \end{array} \qquad \begin{array}{r|rrrr} \frac{5}{3} & 3 & -11 & 2 & 20 \\ & & 5 & -10 & 20 \\ \hline & 3 & -6 & -12 & 0 \end{array} \longleftarrow \frac{5}{3} \text{ is a zero}$$

So $g(x) = \left(x - \frac{5}{3}\right)(3x^2 - 6x - 12) = (3x - 5)(x^2 - 2x - 4)$.

It follows that $f(x) = (2x - 1)(3x^3 - 11x^2 - 2x + 20) = (2x - 1)(3x - 5)(x^2 - 2x - 4)$

STEP 6: Find the remaining zeros of f by solving $x^2 - 2x - 4 = 0$.

$x = \dfrac{2 \pm \sqrt{(-2^2) - 4(1)(-4)}}{2}$ **Substitute into the quadratic formula.**

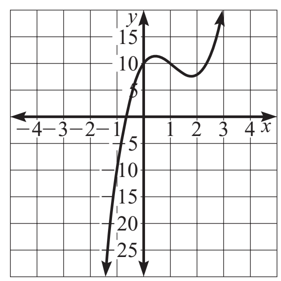
$x = \dfrac{2 \pm \sqrt{20}}{2} = \dfrac{2 \pm 2\sqrt{5}}{2} = 1 \pm \sqrt{5}$ **Simplify.**

The zeros of f are $\frac{1}{2}, \frac{5}{3}, 1 + \sqrt{5}$, and $1 - \sqrt{5}$.

b. STEP 1: List the possible rational zeros of g:

$\pm 1, \pm 2, \pm 5, \pm 10, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}, \pm\frac{10}{3}$

STEP 2: Choose reasonable values from the list above to check, using the graph of the function. For g, the value $x = -\frac{2}{3}$ is reasonable based on the graph shown at the right.



STEP 3: Check the value using synthetic division.

$$\begin{array}{r|rrrr} -\frac{2}{3} & 3 & -10 & 7 & 10 \\ & & -2 & 8 & -10 \\ \hline & 3 & -12 & 15 & 0 \end{array} \longleftarrow -\frac{2}{3} \text{ is a zero}$$

STEP 4: Factor out a binomial using the result of the synthetic division.

$g(x) = \left(x + \frac{2}{3}\right)(3x^2 - 12x + 15)$ **Write as a product of factors.**

$= \left(x + \frac{2}{3}\right)(3)(x^2 - 4x + 5)$ **Factor 3 out of the second factor.**

$= (3x + 2)(x^2 - 4x + 5)$ **Multiply the first factor by 3.**

STEP 5: Find the remaining zeros of g by solving $x^2 - 4x + 5 = 0$.

$x = \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(5)}}{2}$ **Substitute into the quadratic formula.**

$x = \dfrac{4 \pm \sqrt{-4}}{2} = \dfrac{4 \pm 2i}{2} = 2 \pm i$ **Simplify.**

The zeros of g are $\frac{2}{3}$, $2 + i$, and $2 - i$.

Notice that our irrational and complex conjugate zeros are a natural consequence of the quadratic formula.

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BENCHMARK 3 C. Factoring Polynomials

PRACTICE **Find all zeros of the function.**

10. $f(x) = 4x^4 - 5x^2 + 42x - 20$ **11.** $g(x) = 3x^3 - 4x^2 + 27x - 36$

5. Use Zeros to Write a Polynomial Function

EXAMPLE **Write a polynomial function *f* of least degree that has rational coefficients, a leading coefficient of 1, and −5 and 3 − 2*i* as zeros.**

Because the coefficients are rational and $3 - 2i$ is a zero, $3 + 2i$ must also be a zero. Use the three zeros and the factor theorem to write $f(x)$ as a product of three factors.

You can check your final answer by evaluating $f(x)$ when $x = -5$, $3 - 2i$, and $3 + 2i$. Be careful when raising the complex zeros to the second and third powers.

$f(x) = [x - (-5)][x - (3 - 2i)][x - (3 + 2i)]$	**Write $f(x)$ in factored form.**
$= (x + 5)[(x - 3) + 2i][(x - 3) - 2i]$	**Regroup terms.**
$= (x + 5)[(x - 3)^2 - (2i)^2]$	**Multiply.**
$= (x + 5)[x^2 - 6x + 9 - 4i^2]$	**Expand binomial.**
$= (x + 5)[x^2 - 6x + 9 + 4]$	**Evaluate i^2.**
$= (x + 5)(x^2 - 6x + 13)$	**Simplify.**
$= x^3 - 6x^2 + 13x + 5x^2 - 30x + 65$	**Multiply.**
$= x^3 - x^2 - 17x + 65$	**Combine like terms.**

PRACTICE **12.** Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and 2, 2 and $1 - \sqrt{2}$ as zeros.

Quiz

Factor the polynomial completely.

1. $8y^3 - 216$ **2.** $81z^4 - 1296$ **3.** $x^3 - 4x^2 - 16x + 64$

4. Factor $g(x) = x^3 - x^2 + 14x + 24$ completely given that $x - 3$ is a factor.

5. Factor $f(x) = 6x^3 - 7x^2 - 56x - 48$ completely given that $x - 4$ is a factor.

List the possible rational zeros of *f* using the rational zero theorem.

6. $f(x) = 6x^4 - 15x^3 - 19x^2 - 5x - 7$

7. $h(x) = 6x^8 - 11x^6 + 15x^4 - 22x^2 + 6$

Find all zeros of the function.

8. $g(x) = x^3 + x^2 - 4x + 6$

9. $f(x) = 3x^4 - 14x^3 + 11x^2 - 14x + 8$

10. Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and 2, $1 + \sqrt{3}$, and $2 - i$ as zeros.

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BENCHMARK 3
(Chapters 5 and 6)

D. Rational Exponents and Radicals

When an exponent is a natural number, it simply indicates a repeated multiplication. By extensive exposure, we become accustomed to identifying perfect squares, and even perfect cubes. In this section, we will focus on exponents that are non-natural, rational numbers. This develops into a fundamental understanding of roots. Radical notation and rational exponents are parallel notations for evaluating *n*th roots, and each is important. They can be interchanged, but in certain situations one may be simpler than the other.

1. Evaluate *n*th Roots and Rational Exponents

Vocabulary

n*th root of *a In general, for an integer n greater than 1, if $b^n = a$, then b is an nth root of a. An nth root of a is written as $\sqrt[n]{a}$ where n is the index of the radical. An alternate and equally important notation for the nth root of a is $a^{1/n}$.

Rational exponents Let $a^{1/n}$ be an nth root of a, and let m be a positive integer. Then $a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m$ and $a^{-m/n} = \frac{1}{a^{m/n}} = \frac{1}{(a^{1/n})^m} = \frac{1}{(\sqrt[n]{a})^m}$, $a \neq 0$

EXAMPLE **Find the indicated real *n*th root(s) of *a*.**

a. $n = 5, a = -32$ **b.** $n = 4, a = -81$

Recall that the *n*th root of 0 is always zero, regardless of what *n* is.

Solution:

a. Because $n = 3$ is odd and $a = -32 < 0$, -32 has one real 5th root. Because $(-2)^5 = -32$, you can write $\sqrt[5]{-32} = -2$ or $(-32)^{1/5} = -2$.

b. Because $n = 4$ is even and $a = -81 < 0$, -81 has no real 4th roots.

EXAMPLE **Evaluate**

a. $(-125)^{2/3}$ **b.** $49^{-3/2}$

Solution:

If there is no index on the radical, as in part b, then $n = 2$.

	Rational Exponent Form	Radical Form
a.	$(-125)^{2/3} = \left((-125)^{1/3}\right)^2 = (-5)^2 = 25$	$(-125)^{2/3} = \left(\sqrt[3]{-125}\right)^2 = (-5)^2 = 25$
b.	$49^{-3/2} = \frac{1}{49^{3/2}} = \frac{1}{(49^{1/2})^3} = \frac{1}{7^3} = \frac{1}{343}$	$49^{-3/2} = \frac{1}{49^{3/2}} = \frac{1}{(\sqrt{49})^3} = \frac{1}{7^3} = \frac{1}{343}$

PRACTICE **Find the indicated real *n*th root(s) of *a*.**

1. $n = 4, a = 81$ **2.** $n = 7, a = 1$ **3.** $n = 10, a = 0$

Evaluate

4. $(64)^{2/3}$ **5.** $(-25)^{5/2}$ **6.** $(-243)^{-4/5}$

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BENCHMARK 3 — D. Rational Exponents and Radicals

BENCHMARK 3

(Chapters 5 and 6)

2. Solve Equations Using *n*th Roots

EXAMPLE **Solve the equation. Round the result to two decimal places when appropriate.**

a. $3x^3 = -648$ **b.** $(r + 4)^6 = 38$

Solution:

a. $3x^3 = -648$

$x^3 = -216$ — **Divide each side by 3.**

$x = \sqrt[3]{-216}$ — **Take 3rd root of each side.**

$x = -6$ — **Simplify.**

b. $(r + 4)^6 = 38$

$r + 4 = \pm\sqrt[6]{38}$ — **Take 6th roots of each side.**

$r = \pm\sqrt[6]{38} - 4$ — **Subtract 4 from each side.**

$r = \sqrt[6]{38} - 4$ or $r = -\sqrt[6]{38} - 4$ — **Write solutions separately.**

$r \approx -2.17$ or $r \approx -5.83$ — **Use a calculator.**

When evaluating $\sqrt[6]{38}$ with a calculator, use the rational exponent form $38^{1/6}$.

PRACTICE **Solve the equation. Round the result to two decimal places when appropriate.**

7. $x^4 = 16$ **8.** $5x^3 = -135$ **9.** $(x - 1)^5 = 100$

3. Use Properties of Radicals

Vocabulary **Like radicals** Two radical expressions with the same index and radicand.

EXAMPLE **Use the properties of radicals to simplify the following expressions.**

a. $\sqrt[5]{4} \cdot \sqrt[5]{-8}$ **b.** $\sqrt[3]{\frac{64a^3}{b^9}}$ **c.** $3\sqrt{5} + 7\sqrt{5} - \sqrt{5}$

Solution:

a. $\sqrt[5]{4} \cdot \sqrt[5]{-8} = \sqrt[5]{4 \cdot (-8)} = \sqrt[5]{-32} = -2$ — **Product property**

b. $\sqrt[3]{\frac{64a^3}{b^9}} = \frac{\sqrt[3]{64a^3}}{\sqrt[3]{b^9}} = \frac{\sqrt[3]{4^3a^3}}{\sqrt[3]{(b^3)^3}} = \frac{\sqrt[3]{4^3} \cdot \sqrt[3]{a^3}}{b^3} = \frac{4a}{b^3}$ — **Product and quotient properties**

c. $3\sqrt{5} + 7\sqrt{5} - \sqrt{5} = (3 + 7 - 1)\sqrt{5} = 9\sqrt{5}$ — **Distributive property**

In part c, the last term $-\sqrt{5}$ should be considered as $-1\sqrt{5}$.

PRACTICE **Use the properties of radicals to simplify the expressions.**

10. $\sqrt[3]{32} \cdot \sqrt[3]{2}$ **11.** $\sqrt{18x^3} \cdot \sqrt{2x}$ **12.** $\frac{\sqrt[4]{48}}{\sqrt[4]{3}}$

13. $\frac{\sqrt[5]{486x}}{\sqrt[5]{2x^{11}}}$ **14.** $-7\sqrt[3]{15} + 8\sqrt[3]{15}$

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BENCHMARK 3
(Chapters 5 and 6)

4. Write Radicals in Simplest Form

Vocabulary

Simplest form A radical with index n if the radicand has no perfect nth powers as factors and any denominator has been rationalized.

EXAMPLE **Write the expression in simplest form.**

a. $\sqrt{243}$ **b.** $\dfrac{\sqrt[3]{10}}{\sqrt[3]{6a}}$

Solution:

a. $\sqrt{243} = \sqrt{81 \cdot 3}$ **Factor out perfect square.**

$= \sqrt{81} \cdot \sqrt{3}$ **Product property**

$= 3\sqrt{3}$ **Simplify.**

b. $\dfrac{\sqrt[3]{10}}{\sqrt[3]{6a}} = \dfrac{\sqrt[3]{10}}{\sqrt[3]{6a}} \cdot \dfrac{\sqrt[3]{36a^2}}{\sqrt[3]{36a^2}}$ **Make denominator a perfect cube.**

$= \dfrac{\sqrt[3]{360a^2}}{\sqrt[3]{216a^3}}$ **Product property**

$= \dfrac{\sqrt[3]{8 \cdot 45a^2}}{6a}$ **Factor out perfect cube.**

$= \dfrac{\sqrt[3]{8} \cdot \sqrt[3]{45a^2}}{6a}$ **Product property**

$= \dfrac{2 \cdot \sqrt[3]{45a^2}}{6a} = \dfrac{\sqrt[3]{45}}{3a}$ **Simplify.**

Notice that in part b we could have applied the quotient property at the very beginning. Our expression would have simplified as $\dfrac{\sqrt[3]{10}}{\sqrt[3]{6a}} = \sqrt{\dfrac{10}{6a}} = \sqrt{\dfrac{5}{3a}} = \dfrac{\sqrt{5}}{\sqrt{3a}}$. Doing so would have eliminated the simplification at the end.

PRACTICE **Write the expression in simplest form.**

15. $\sqrt[3]{-250}$ **16.** $\sqrt{363x^6y^7}$ **17.** $\dfrac{\sqrt[4]{x}}{\sqrt[4]{8x^2}}$

Quiz

1. Find the real 4th root(s) of 1294.

2. Evaluate $(25)^{-3/2}$ without using a calculator.

Solve the equation. Round the result to two decimal places when appropriate.

3. $x^4 = 256$ **4.** $4x^5 = 12{,}500$ **5.** $(x - 2)^5 = -81$

Simplify the expression.

6. $\sqrt[3]{-250}$ **7.** $-15\sqrt[3]{6} + 2\sqrt[3]{48}$ **8.** $\dfrac{\sqrt{150xy^5}}{\sqrt{36x^3y^2}}$

9. $\sqrt{18x^3} \cdot \sqrt{2x}$ **10.** $\dfrac{\sqrt[4]{48}}{\sqrt[4]{3}}$ **11.** $\dfrac{\sqrt[5]{486x}}{\sqrt[5]{2x^{11}}}$

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BENCHMARK 3

(Chapters 5 and 6)

E. Function Operations and Composition

Once the general idea of a function is understood, we can begin to develop a more sophisticated study of mathematics. In this section, the process of creating new functions based on existing functions is studied.

1. Perform Operations on Functions

Vocabulary

Operations on Functions Let f and g be any two functions. A new function h can be defined by performing any of the four basic operations of addition, subtraction, multiplication, or division on f and g.

EXAMPLE **Let $f(x) = 5x^{3/4}$ and $g(x) = 2x^{3/4} + 7$. Find the following.**

a. $f(x) + g(x)$ **b.** $f(x) - g(x)$ **c.** the domains of $f + g$ and $f - g$

Solution:

a. $f(x) + g(x) = 5x^{3/4} + (-2x^{3/4} + 7)$

$= (5 + (-2))\,x^{3/4} + 7 = 3x^{3/4} + 7$

Notice that $f + g$ and $g + f$ are equal, while $f - g$ and $g - f$ are *not* equal.

b. $f(x) - g(x) = 5x^{3/4} - (-2x^{3/4} + 7)$ **Be sure to distribute the −1.**

$= (5 - (-2))\,x^{3/4} - 7 = 7x^{3/4} - 7$

c. The functions f and g each have the same domain: all nonnegative real numbers. So, the domains of $f + g$ and $f - g$ also consist of all nonnegative reals.

EXAMPLE **Let $f(x) = -6x^3$ and $g(x) = 3x^{1/3}$. Find the following.**

a. $f(x) \cdot g(x)$ **b.** $\frac{f(x)}{g(x)}$ **c.** the domains of $f \cdot g$ and $\frac{f}{g}$

Solution:

a. $f(x) \cdot g(x) = (-6x^3)3x^{1/3} = -18x^{(3 + 1/3)} = -18x^{10/3}$

Notice that $f \cdot g$ and $g \cdot f$ are equal, while $\frac{f}{g}$ and $\frac{g}{f}$ are *not* equal.

b. $\frac{f(x)}{g(x)} = \frac{-6x^3}{3x^{1/3}} = \frac{-6}{3} \cdot \frac{x^3}{x^{1/3}} = -2x^{(3 - 1/3)} = -2x^{8/3}$

c. The functions f and g each have the same domain: all real numbers. So, the domain of $f \cdot g$ consists of all real numbers. Because $g(0) = 0$, the domain of $\frac{f}{g}$ is restricted to all real numbers except 0.

PRACTICE **Let $f(x) = 5x^{-1/2}$, $g(x) = -3x^{-1/2}$, and $h(x) = x^2 - 4x$. Perform the indicated operation and state the domain.**

1. $f(x) - g(x)$ **2.** $g(x) - h(x)$ **3.** $\frac{f(x)}{g(x)}$ **4.** $\frac{h(x)}{f(x)}$

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2. Find Compositions of Functions

Vocabulary

Composition of Functions The *composition* of a function g with a function f is $h(x) = g(f(x))$. The domain of h is the set of all x-values such that x is in the domain of f and $f(x)$ is in the domain of g.

EXAMPLE **Let $f(x) = \sqrt{x - 2}$ and $g(x) = x^2 + 1$. Find the composition and its domain.**

a. $f(g(x))$ **b.** $g(f(x))$ **c.** $g(g(x))$

Do not base the domain on the final, simplified composition. The composition's domain can be no larger than the domain of the inner function. Then determine what values, if any, should not be included based on the domain of the outer function.

Solution:

a. $f(g(x)) = f(x^2 + 1) = \sqrt{(x^2 + 1) - 2} = \sqrt{x^2 - 1}$

The domain of $f(g(x))$ is $(-\infty, -1] \cup [1, \infty)$. The domain of g is all real numbers, but the numbers in the interval $(-1, 1)$ have images that are not in the domain of f.

b. $g(f(x)) = g\left(\sqrt{x - 2}\right) = \left(\sqrt{x - 2}\right)^2 + 1 = (x - 2) + 1 = x - 1$

The domain of $g(f(x))$ is $[2, \infty)$. It can not be any larger than the domain of f.

c. $g(g(x)) = g(x^2 + 1) = (x^2 + 1)^2 + 1 = x^4 + 2x^2 + 1 + 1 = x^4 + 2x^2 + 2$

The domain of $g(g(x))$ is all real numbers.

PRACTICE **Let $f(x) = \frac{1}{x - 2}$, $g(x) = x^{-1}$, and $h(x) = \frac{1}{x} + 2$. Find the composition and its domain.**

5. $g(f(x))$ **6.** $h(g(x))$ **7.** $f(h(x))$

3. Find the Inverse of a Function

Vocabulary

Inverse Functions Functions f and g are inverses of each other provided $f(g(x)) = x$ and $g(f(x)) = x$. The function g is denoted by f^{-1}, read as "f inverse."

Horizontal Line Test The inverse of a function f is also a function if and only if no horizontal line intersects the graph of f more than once.

EXAMPLE **Consider the function. Determine whether its inverse is a function, and then find the inverse.**

a. $f(x) = \frac{1}{2}x - 3$ **b.** $g(x) = (x - 1)^2$

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Solution:

a. Graph the function f. No horizontal line intersects the graph more than once. So the inverse is itself a function.

$f(x) = \frac{1}{2}x - 3$	**Write function.**
$y = \frac{1}{2}x - 3$	**Replace $f(x)$ with y.**
$x = \frac{1}{2}y - 3$	**Switch x and y.**
$x + 3 = \frac{1}{2}y$	**Add 3 to each side.**
$2x + 6 = y$	**Mult. each side by 2.**
$f^{-1}(x) = 2x + 6$	**Rewrite as a function.**

b. Graph the function g. The horizontal line $y = 2$ intersects the graph twice. The inverse is not a function.

$g(x) = (x - 1)^2$	**Write function.**
$y = (x - 1)^2$	**Replace $g(x)$ with y.**
$x = (y - 1)^2$	**Switch x and y.**
$\pm\sqrt{x} = y - 1$	**Square root each side.**
$\pm\sqrt{x} + 1 = y$	**Add 1 to each side.**
$y = \pm\sqrt{x} + 1$	**Rewrite.**
$\pm\sqrt{x} + 1 = y$	

Because the inverse of $g(x)$ in example b is not a function, we can not use the inverse notation $g^{-1}(x)$.

PRACTICE **Consider the function. Determine whether its inverse is a function, and then find the inverse.**

8. $f(x) = 5x - 1$ **9.** $g(x) = -x^2 + 4, x \geq 0$ **10.** $h(x) = x^3 + 8$

4. Use the Inverse of a Power Function

Vocabulary **Power Function** A power function is a common function which has the form $y = ax^b$ where a is a real number and b is a rational number.

EXAMPLE **The average weight *W* (in pounds) of SUVs produced in the U.S. can be modeled by $W = 3480t^{0.09}$ where *t* is the number of years since 1985. Find the inverse model that gives time as a function of the average SUV weight, and use it to predict when the average weight will reach 4750 pounds.**

In this example, we round $\frac{1}{0.09}$ to 11.1. If you can use more powerful calculators, it is not necessary to round at this point. In general, the longer you delay rounding, the more accurate your answers will be.

Solution:

$W = 3480t^{0.09}$	**Write original model.**
$\frac{W}{3480} = t^{0.09}$	**Divide each side by 3480.**
$\left(\frac{W}{3480}\right)^{\frac{1}{0.09}} = (t^{0.09})^{1/0.009}$	**Raise each side to the power $\frac{1}{0.09}$.**
$\left(\frac{W}{3480}\right)^{11.1} = t$	**Simplify. This is the inverse model.**
$t = \left(\frac{4750}{3480}\right)^{11.1} \approx 30.6$	**Set $W = 4750$ in the model.**

The average weight will reach 4750 pounds about 30.6 years after 1985, or sometime in 2015.

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PRACTICE

The average weight W (in pounds) of compact cars produced in the U.S. can be modeled by $W = 2150t^{0.05}$, where t is the number of years since 1985.

11. Find the inverse model that gives time as a function of the average compact car weight.

12. Predict when the average weight will reach 2500 pounds.

Quiz

Let $f(x) = 6x^{1/2}$, $g(x) = 2x^{1/4}$, and $h(x) = -3x^{1/2}$. Perform the indicated operation and state the domain.

1. $f(x) + h(x)$ **2.** $f(x) \cdot g(x)$ **3.** $\frac{g(x)}{h(x)}$

Let $f(x) = 2x + 1$, $g(x) = \frac{x - 4}{3}$, and $h(x) = -x^2 + 5$. Find the composition and its domain.

4. $f(g(x))$ **5.** $g(h(x))$ **6.** $h(f(x))$

Consider the function. Determine whether its inverse is a function, and then find the inverse.

7. $f(x) = -10x + 7$ **8.** $g(x) = 3x^{1/5}$ **9.** $h(x) = \frac{2x^3}{3}$

The average size S (in cubic centimeters) of tomatoes produced at a California farm can be modeled by $S = 110t^{0.02}$, where t is the number of years since 1990.

10. Find the inverse model that gives time as a function of the average tomato size.

11. Predict when the average size will reach 140 cubic centimeters.

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F. Radical Equations

Once a solid understanding of radicals is achieved, we can begin working with these functions. Graphing our new functions is a logical progression in our mathematical development. After this, we will of course wish to solve equations containing radicals. To do so, we will need some new tools, and checking our solutions becomes mandatory. Once these techniques have been mastered, we will have increased dramatically the types of equations we are capable of solving.

1. Graph a Radical Function

Vocabulary

Parent functions The parent function for the family of square root functions is $f(x) = \sqrt{x}$, and for the family of cube root functions is $g(x) = \sqrt[3]{x}$.

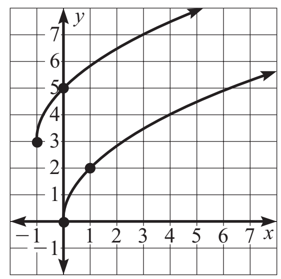
EXAMPLE **Graph the function. Then state the domain and range.**

a. $y = 2\sqrt{x + 1} + 3$ **b.** $y = -4\sqrt[3]{x} - 4$

Solution:

The horizontal translation is the opposite signed value of the number in the radical. The vertical translation is the same sign and value as the constant term.

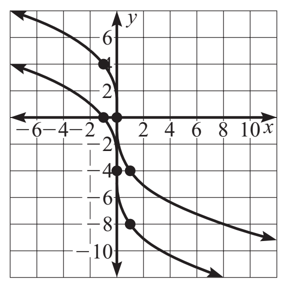
a. **STEP 1:** Sketch the graph of $y = 2\sqrt{x}$ (the lighter graph). Notice that it begins at the origin and passes through the point (1, 2).



STEP 2: Translate the graph. For $y = 2\sqrt{x + 1} + 3$, $h = -1$, and $k = 3$. So, shift the graph left 1 unit and up 3 units. The resulting graph starts at $(-1, 3)$ and passes through (0, 5).

From the graph you can see that the domain of the function is $x \geq -1$ and the range is $y \geq 3$.

b. **STEP 1:** Sketch the graph of $y = -4\sqrt[3]{x}$ (the lighter graph). Notice that it passes through the origin and the points $(1, -4)$, and $(-1, 4)$.



In example b, we know x is the same as $x - 0$, so the horizontal translation is 0.

STEP 2: Translate the graph. For $y = -4\sqrt[3]{x} - 4$, $h = 0$, and $k = -4$. So, shift do not shift the graph horizontally and down 4 units. The resulting graph passes through the points $(0, -4)$, $(1, -8)$, and $(-1, 0)$.

From the graph you can see that the domain and range of the function are both all real numbers.

PRACTICE **Graph the function. Then state the domain and range.**

1. $y = \sqrt{x - 3} + 2$ **2.** $y = 2\sqrt[3]{x + 1}$ **3.** $y = -5\sqrt{x + 3} + 2$

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BENCHMARK 3

(Chapters 5 and 6)

2. Solve a Radical Equation

Vocabulary **Radical equations** Equations with radicals that have variables in their radicands.

EXAMPLE **Solve $2\sqrt{3x - 5} = 14$. Check the solution.**

Solution:

When the radical equation contains a cube root, cubing each side will eliminate an isolated radical. This can be extended to any *n*th root.

$2\sqrt{3x - 5} = 14$	**Write original equation.**
$\sqrt{3x - 5} = 7$	**Isolate the radical.**
$\left(\sqrt{3x - 5}\right)^2 = (7)^2$	**Square each side to eliminate the radical.**
$3x - 5 = 49$	**Simplify.**
$3x = 54$	**Add 5 to each side.**
$x = 18$	**Divide each side by 3.**

CHECK Check $x = 18$ in the original equation.

$2\sqrt{3(18) - 5} \stackrel{?}{=} 14$	**Substitute 18 for *x*.**
$2\sqrt{49} \stackrel{?}{=} 14$	**Simplify.**
$14 = 14$ ✓	**Solution checks.**

PRACTICE **Solve the equation. Check your solutions.**

4. $\sqrt{2x - 7} = 5$ **5.** $\frac{\sqrt{-2x}}{5} + 1 = 3$ **6.** $3\sqrt[3]{x + 1} - 4 = 2$

3. Solve an Equation with Extraneous Solutions

Vocabulary **Extraneous solution** Appears to be a solution, but does not actually yield true statements when substituted into the original equation.

EXAMPLE **Solve $2x + 3 = \sqrt{4x + 14}$.**

Solution:

$2x + 3 = \sqrt{4x + 14}$	**Write original equation.**
$(2x + 3)^2 = \left(\sqrt{4x + 14}\right)^2$	**Square each side.**
$4x^2 + 12x + 9 = 4x + 14$	**Expand left side and simplify right side.**
$4x^2 + 8x - 5 = 0$	**Write in standard form.**
$(2x + 5)(2x - 1) = 0$	**Factor left side.**
$2x + 5 = 0$ or $2x - 1 = 0$	**Zero-product property**
$x = -\frac{5}{2}$ or $x = \frac{1}{2}$	**Solve for *x*.**

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BENCHMARK 3

(Chapters 5 and 6)

The directions for this problem did not tell us to check our solutions. It was only by checking, however, that we were able to determine one of the apparent solutions was extraneous. *Always* check your apparent solution(s) of radical equations.

CHECK

Check $x = -\frac{5}{2}$ in original equation.

$2\left(-\frac{5}{2}\right) + 3 \stackrel{?}{=} \sqrt{4\left(-\frac{5}{2}\right) + 14}$

$-5 + 3 \stackrel{?}{=} \sqrt{-10 + 14}$

$-2 \stackrel{?}{=} \sqrt{4}$

$-2 \neq 2$

Check $x = \frac{1}{2}$ in original equation.

$2\left(\frac{1}{2}\right) + 3 \stackrel{?}{=} \sqrt{4\left(\frac{1}{2}\right) + 14}$

$1 + 3 \stackrel{?}{=} \sqrt{2 + 14}$

$4 \stackrel{?}{=} \sqrt{16}$

$4 = 4$ ✓

The only solution is $x = \frac{1}{2}$. (The apparent solution $x = -\frac{5}{2}$ is extraneous.)

PRACTICE **Solve the equation. Check for extraneous solutions.**

7. $x = \sqrt{5x - 6}$

8. $x = \sqrt{-5x - 6}$

9. $\sqrt{18x + 45} - 5 = x$

10. $\sqrt[3]{7x + 13} = x + 1$

Quiz

Graph the function. Then state the domain and range.

1. $y = -3\sqrt{x + 4} - 1$

2. $y = -2\sqrt[3]{x - 1} + 2$

3. $y = 4\sqrt{x - 5} + 4$

Solve the equation. Check your solutions.

4. $2\sqrt{x - 3} + 1 = 7$

5. $\sqrt[3]{3x - 5} = 4$

6. $\frac{\sqrt{5x + 11}}{4} = \frac{3}{2}$

7. $\sqrt{9x - 3} = 3x - 1$

8. $x + 2 = \sqrt{6x + 19}$

9. $\sqrt{x + 24} = x + 4$