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BENCHMARK 6

(Chapters 13 and 14)

A. Trigonometric Functions (pp. 118–123)

Ratios of the sides of a right triangle are used to define the six trigonometric functions. These trigonometric functions, in turn, are used to help find unknown side lengths and angle measures of triangles.

1. Evaluate Trigonometric Functions

Vocabulary

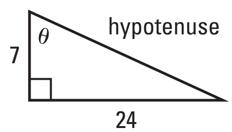
The six trigonometric functions and their abbreviations are:
sine, or sin
cosine, or cos
tangent, or tan
cotangent, or cot
secant, or sec
cosecant, or csc

In a right triangle with acute angle θ, the six trigonometric functions are defined as follows:

$\sin\theta = \dfrac{\text{opposite side}}{\text{hypotenuse}}$ $\qquad \cos\theta = \dfrac{\text{adjacent side}}{\text{hypotenuse}}$ $\qquad \tan\theta = \dfrac{\text{opposite side}}{\text{adjacent side}}$

$\csc\theta = \dfrac{\text{hypotenuse}}{\text{opposite side}}$ $\qquad \sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent side}}$ $\qquad \cot\theta = \dfrac{\text{adjacent side}}{\text{opposite side}}$

EXAMPLE **Evaluate the six trigonometric functions of the angle θ.**



Solution:

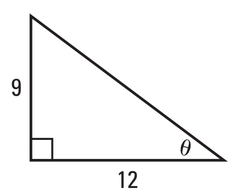
From the Pythagorean Theorem, the hypotenuse has length $\sqrt{7^2 + 24^2} = \sqrt{625} = 25$.

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{24}{25}$ $\qquad \cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{7}{25}$ $\qquad \tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{24}{7}$

$\csc\theta = \dfrac{\text{hyp}}{\text{opp}} = \dfrac{25}{24}$ $\qquad \sec\theta = \dfrac{\text{hyp}}{\text{adj}} = \dfrac{25}{7}$ $\qquad \cot\theta = \dfrac{\text{adj}}{\text{opp}} = \dfrac{7}{24}$

PRACTICE **Evaluate the six trigonometric functions of the angle θ.**

1.



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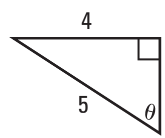

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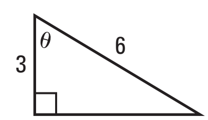

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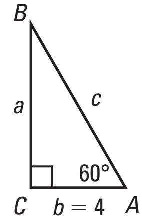
2. Solve a Right Triangle

EXAMPLE **Solve $\triangle ABC$.**



Solution:

A and B are complementary angles, so $B = 90° - 60° = 30°$.

$\csc \theta = \frac{1}{\sin \theta}$

$\sec \theta = \frac{1}{\cos \theta}$

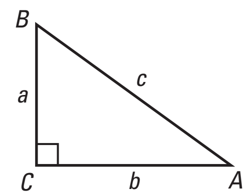
$\cot \theta = \frac{1}{\tan \theta}$

$\tan 60° = \frac{\text{opp}}{\text{adj}}$	$\sec 60° = \frac{\text{hyp}}{\text{adj}}$	**Write trigonometric equation.**
$\tan 60° = \frac{a}{4}$	$\sec 60° = \frac{c}{4}$	**Substitute.**
$a = 4(\tan 60°)$	$c = 4(\sec 60°) = 4\left(\frac{1}{\cos 60°}\right)$	**Solve for the variable.**
$a = 4\sqrt{3}$	$c = 4\left(\frac{1}{\frac{1}{2}}\right) = 4 \cdot 2 = 8$	**Evaluate trigonometric functions.**

PRACTICE **Solve $\triangle ABC$ using the diagram and given measurements.**

4. $B = 45°, c = 12\sqrt{2}$

5. $A = 30°, b = 6$



3. Draw General Angles

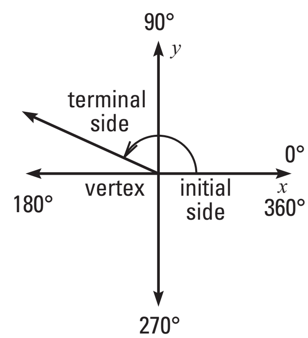
Vocabulary

The measure of an angle is positive if the rotation of the terminal side is counterclockwise. If the rotation is clockwise, the angle measure is negative.

Initial side The fixed ray of an angle on a coordinate plane.

Terminal side The ray of an angle that is rotated about the vertex on a coordinate plane.

Standard position The location of an angle whose vertex is at the origin and initial side lies on the positive x-axis.

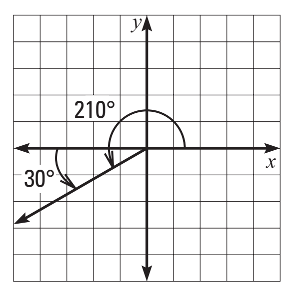


EXAMPLE **Draw an angle with the given measure in standard position.**

a. 210° **b.** 480° **c.** −135°

Solution:

a. Since 210° is 30° more than 180°, the terminal side is 30° counterclockwise past the negative x-axis.



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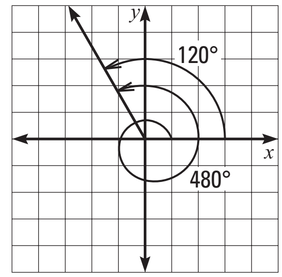
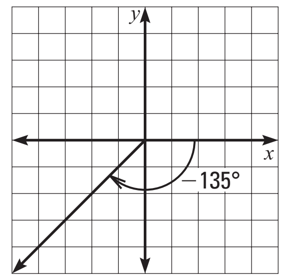
b. Since 480° is 120° more than 360°, the terminal side makes one whole revolution counterclockwise plus 120° more.



c. Since –135° is negative, the terminal side is 135° clockwise from the positive x-axis.



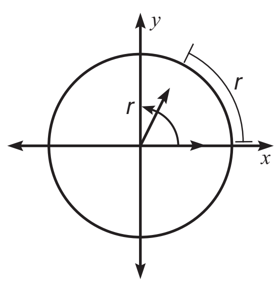
PRACTICE **Draw an angle with the given measure in standard position.**

6. 300° **7.** −225° **8.** 600°

4. Convert Between Degrees and Radians

Vocabulary

Radian The measure of an angle in standard position whose terminal side intercepts an arc of length r.



To convert degrees to radians, multiply degrees by $\frac{\pi \text{ radians}}{180°}$.

To convert radians to degrees, multiply radians by $\frac{180°}{\pi \text{ radians}}$.

EXAMPLE **Convert (a) 225° to radians and (b) $\frac{2\pi}{3}$ to degrees.**

Solution:

a. $225° = 225° \cdot \frac{\pi \text{ radians}}{180°} = \frac{5\pi}{4}$ radians, or simply $\frac{5\pi}{4}$

b. $\frac{2\pi}{3} = \frac{2\pi}{3}$ radians $\cdot \frac{180°}{\pi \text{ radians}} = 120°$

PRACTICE **Convert the degree measure to radians or the radian measure to degrees.**

9. 150° **10.** $\frac{7\pi}{6}$ **11.** $\frac{\pi}{9}$

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5. Evaluate Trigonometric Functions Given a Point

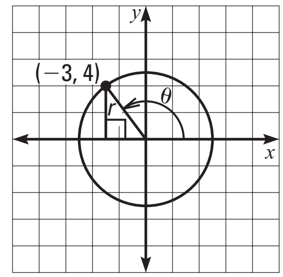
For an angle θ in standard position whose terminal side intersects a circle with radius r, the trigonometric functions are defined as follows:

$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$

$\tan\theta = \frac{y}{x}, x \neq 0$

EXAMPLE **The point (−3, 4) is a point on the terminal side of an angle θ in standard position. Evaluate the six trigonometric functions of θ.**



Solution:

By the Pythagorean Theorem,

$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5.$

For $x = -3$, $y = 4$, and $r = 5$,

$\sin\theta = \frac{y}{r} = \frac{4}{5}$ $\cos\theta = \frac{x}{r} = -\frac{3}{5}$ $\tan\theta = \frac{y}{x} = -\frac{4}{3}$

$\csc\theta = \frac{1}{\sin\theta} = \frac{5}{4}$ $\sec\theta = \frac{1}{\cos\theta} = -\frac{5}{3}$ $\cot\theta = \frac{1}{\tan\theta} = -\frac{3}{4}$

PRACTICE **Evaluate the six trigonometric functions with the given point on the terminal side of angle θ.**

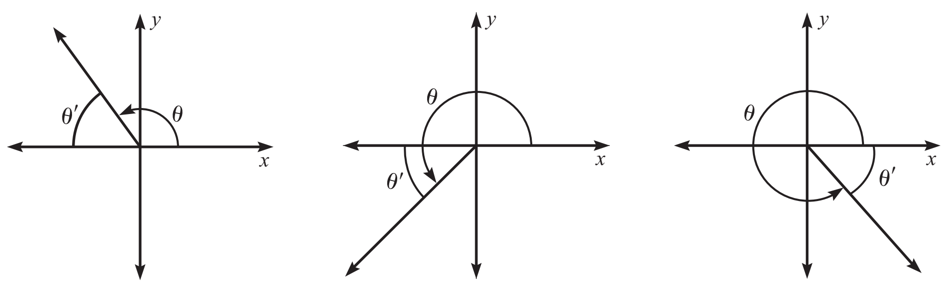
12. (6, 8) **13.** (8, −15) **14.** (−12, −5)

6. Use Reference Angles

Vocabulary **Reference angle** The acute angle θ' formed by the terminal side of an angle θ in the standard position and the x-axis.

Signs of trigonometric functions in each quadrant are as follows:

Quadrant I
$\sin\theta = +$
$\cos\theta = +$
$\tan\theta = +$



Quadrant II
$\sin\theta = +$
$\cos\theta = -$
$\tan\theta = -$

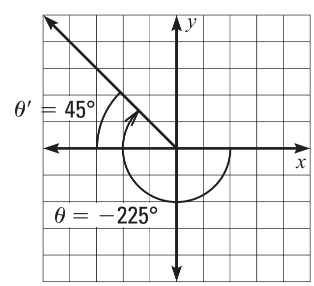
Quadrant III
$\sin\theta = -$
$\cos\theta = -$
$\tan\theta = +$

Quadrant IV
$\sin\theta = -$
$\cos\theta = +$
$\tan\theta = -$

EXAMPLE **Use reference angles to evaluate (a) tan (−225°) and (b) cos $\frac{8\pi}{3}$.**

Solution:

a. The angle −225° is coterminal with 135°.
The reference angle is 180° − 135° = 45°.
The tangent function is negative in Quadrant II, so tan (−225°) = −tan 45° = −1.



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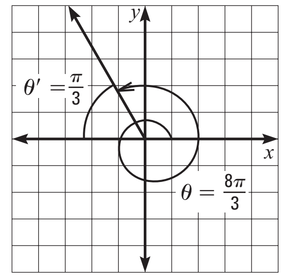
b. The angle $\frac{8\pi}{3}$ is coterminal with $\frac{2\pi}{3}$.

The reference angle is $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$.

The cosine function is negative in Quadrant II,

so $\cos \frac{8\pi}{3} = -\cos \frac{\pi}{3} = -\frac{1}{2}$.



PRACTICE **Evaluate the trigonometric function without using a calculator.**

15. $\cos 300°$ **16.** $\csc -330°$ **17.** $\tan \frac{17\pi}{6}$

7. Evaluate Inverse Trigonometric Functions, Solve a Trigonometric Equation

For $\sin \theta = a$, the **inverse sine** is $\sin^{-1} a = \theta$, $-90° \le \theta \le 90°$.

For $\cos \theta = a$, the **inverse cosine** is $\cos^{-1} a = \theta$, $0° \le \theta \le 180°$.

For $\tan \theta = a$, the **inverse tangent** is $\tan^{-1} a = \theta$, $-90° < \theta < 90°$.

EXAMPLE **Evaluate (a) $\sin^{-1} 0.5$ and (b) $\tan^{-1} \sqrt{3}$.**

Solution:

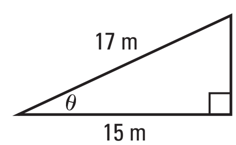
a. When $-90° \le \theta \le 90°$, the angle whose sine is 0.5 is:
$\theta = \sin^{-1} 0.5 = 30°$

b. When $-90° < \theta < 90°$, the angle whose tangent is $\sqrt{3}$ is:
$\theta = \tan^{-1} \sqrt{3} = 60°$

EXAMPLE **A 17-meter ramp has a horizontal length of 15 meters. What is the angle θ of the ramp?**

Solution:

Draw a triangle that represents the ramp. Write a trigonometric equation that involves the ramp's length and horizontal length. $\cos \theta = \frac{15}{17}$



Use a calculator to find the measure of θ.

$$\theta = \cos^{-1} \frac{15}{17} \approx 28.1°$$

PRACTICE **Evaluate the expression.**

18. $\tan^{-1} -1$ **19.** $\cos^{-1} 0.5$ **20.** $\sin^{-1} \sqrt{2}$

21. A cable wire is attached to the top of a 10-foot pole 6 feet from the base of the pole. What is the angle the wire makes with the ground?

BENCHMARK 6 A. Trigonometric Functions

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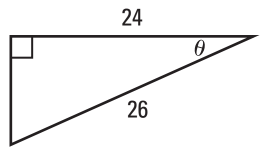
BENCHMARK 6

(Chapters 13 and 14)

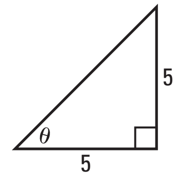
Quiz

Evaluate the six trigonometric functions of the angle θ.

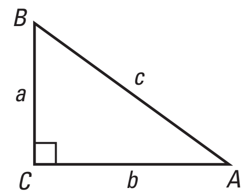
1.



2.



Solve $\triangle ABC$ using the diagram and given measurements.



3. $A = 45°, a = 11$

4. $B = 60°, c = 15$

Convert the degree measure to radians or the radian measure to degrees.

5. $75°$

6. $\frac{11\pi}{3}$

7. $250°$

Evaluate the six trigonometric functions with the given point on the terminal side of angle θ.

8. $(-20, 15)$

9. $(-24, -7)$

10. $(2, -2)$

Evaluate without using a calculator.

11. $\tan(-120°)$

12. $\cos \frac{5\pi}{2}$

13. $\cos^{-1} \frac{\sqrt{2}}{2}$

14. $\sin^{-1} -\frac{\sqrt{3}}{2}$

15. An airplane begins its descent for landing at an altitude of 29,000 feet. At this time, the airplane is 50 miles from the runway. At what angle does the airplane descend?

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BENCHMARK 6
(Chapters 13 and 14)

B. Law of Sines and Law of Cosines (pp. 124–126)

The six trigonometric ratios can be used to solve right triangles. When a triangle contains no right angles, formulas relating to sine and cosine can be used to solve the triangle.

1. Use the Law of Sines

Vocabulary

Law of sines A method for solving a triangle when two angles and a side are known (AAS or ASA cases) or when the lengths of two sides and an angle opposite one of those sides are known (SSA case).

EXAMPLE **Solve $\triangle ABC$ with $A = 112°$, $C = 27°$, and $c = 7$.**

For $\triangle ABC$ with opposite sides a, b, and c, the law of sines is: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Solution:

First find the third angle: $B = 180° - 112° - 27° = 41°$.

By the law of sines, $\frac{\sin 112°}{a} = \frac{\sin 41°}{b} = \frac{\sin 27°}{7}$.

$\frac{\sin 112°}{a} = \frac{\sin 27°}{7}$	**Write two equations with one variable.**	$\frac{\sin 41°}{b} = \frac{\sin 27°}{7}$
$a = \frac{7 \sin 112°}{\sin 27°}$	**Solve for each variable.**	$b = \frac{7 \sin 41°}{\sin 27}$
$a \approx 14.3$	**Use a calculator.**	$b \approx 10.1$

PRACTICE **Solve $\triangle ABC$.**

1. $A = 45°, B = 63°, c = 10$

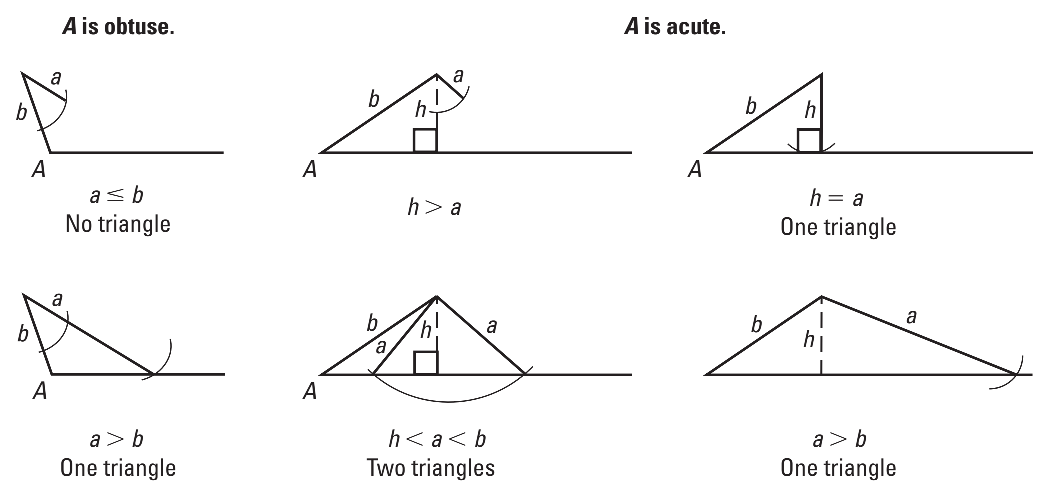
2. $C = 105°, B = 30°, b = 16$

2. Examine SSA Triangles

Vocabulary

SSA case When the lengths of two sides and an angle opposite one of those sides are known, resulting in either no triangle, one triangle, or two different triangles.

In the diagram at right, $h = b \sin A$.



EXAMPLE **Determine the number of triangles that can be formed. Solve the triangle if only one triangle can be formed.**

a. $A = 124°, a = 13, b = 8$

b. $A = 130°, a = 6, b = 10$

c. $A = 55°, a = 12, b = 14$

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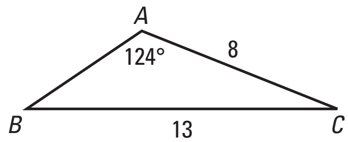
Solution:

a. Since A is obtuse and the side opposite A is longer than the given adjacent side, only one triangle can be formed. Use the law of sines to solve the triangle.



$\frac{\sin 124°}{13} = \frac{\sin B}{8}$ **Law of sines**

$\sin B = \frac{8 \sin 124°}{13} \approx 0.5102$ **Multiply each side by 8.**

$B \approx 30.7°$ **Use inverse sine function.**

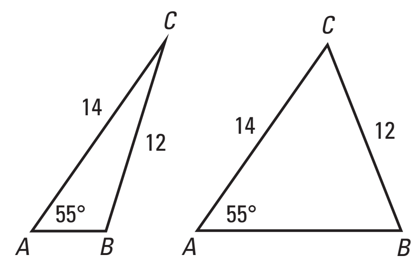
$C \approx 180° - 124° - 30.7° = 25.3°$

$\frac{\sin 124°}{13} = \frac{\sin 25.3°}{c}$ **Law of sines**

$c = \frac{13 \sin 25.3°}{\sin 124°} \approx 6.7$ **Cross multiply.**

b. Since A is obtuse and the side opposite A is shorter than the given adjacent side, it is not possible to draw the indicated triangle. No triangle exists with these given sides and angle.

c. Since $b \sin A = 14 \sin 55° \approx 11.5$, and $11.5 < 12 < 14$ $(h < a < b)$, two triangles can be formed.



Triangle 1 Triangle 2

PRACTICE **Determine the number of triangles that can be formed.**

3. $A = 60°, a = 6, b = 9$

4. $A = 38°, a = 20, b = 25$

5. $A = 143°, a = 18, b = 15$

3. Find an Unknown Side with the Law of Cosines

Vocabulary **Law of cosines** A method for solving a triangle when two sides and the included angle are known (SAS case) or when all three sides are known (SSS case).

EXAMPLE **Find the unknown side in $\triangle ABC$ when $b = 16$, $c = 12$, and $A = 75°$.**

For $\triangle ABC$ with opposite sides a, b, and c, the law of cosines is:
$a^2 = b^2 + c^2 - 2bc \cos A$
$b^2 = a^2 + c^2 - 2ac \cos B$
$c^2 = a^2 + b^2 - 2ab \cos C$

Solution:

$a^2 = b^2 + c^2 - 2bc \cos A$ **Law of cosines**

$a^2 = 16^2 + 12^2 - 2(16)(12) \cos 75°$ **Substitute for *b*, *c*, and *A*.**

$a^2 \approx 300.6$ **Simplify.**

$a \approx \sqrt{300.6} \approx 17.3$ **Take positive square root.**

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(Chapters 13 and 14)

PRACTICE **Find the unknown side in $\triangle ABC$.**

6. $a = 7, c = 5$, and $B = 108°$

7. $a = 24, b = 6$, and $C = 19°$

4. Find Unknown Angles with the Law of Cosines

EXAMPLE **Solve $\triangle ABC$ with $a = 9$, $b = 12$, and $c = 10$.**

Solution:

Find the angle opposite the longest side using the law of cosines.

$b^2 = a^2 + c^2 - 2ac \cos B$ **Law of cosines**

$12^2 = 9^2 + 10^2 - 2(9)(10) \cos B$ **Substitute.**

$\frac{12^2 - 9^2 - 10^2}{-2(9)(10)} = \cos B$ **Solve for cos *B*.**

$0.2056 \approx \cos B$ **Simplify.**

$B \approx 78.1°$ **Use inverse cosine.**

Now use the law of sines.

$\frac{\sin A}{a} = \frac{\sin B}{b}$ **Law of sines**

$\frac{\sin A}{9} = \frac{\sin 78.1°}{12}$ **Substitute.**

$\sin A = \frac{9 \sin 78.1°}{12} \approx 0.7339$ **Multiply each side by 9 and simplify.**

$A \approx 47.2°$ **Use inverse sine.**

The third angle of the triangle is $C \approx 180° - 47.2° - 78.1° = 54.7°$.

PRACTICE **8.** Solve $\triangle ABC$ with $a = 6$, $b = 7$, and $c = 11$.

Quiz

Determine the number of triangles that can be formed.

1. $A = 141°, a = 16, b = 11$

2. $A = 54°, a = 14, b = 17$

3. $A = 99°, a = 8, b = 8$

Solve $\triangle ABC$.

4. $B = 50°, A = 82°, a = 5$

5. $A = 37°, C = 97°, b = 9$

6. $b = 13, c = 18, A = 100°$

7. $a = 4, b = 5, C = 161°$

8. $a = 5, b = 2, c = 4$

9. $a = 20, b = 14, c = 23$

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BENCHMARK 6
(Chapters 13 and 14)

C. Graph Trigonometric Functions (pp. 127–129)

You learned how to use sine, cosine, and tangent functions to solve right triangles. Here you will learn how to graph these functions on the coordinate plane.

1. Graph Sine and Cosine Functions

Vocabulary

Amplitude Half the difference between a function's maximum M and minimum m.

Periodic function A function with a repeating pattern.

Cycle The shortest repeating portion of a graph.

Period The horizontal length of each cycle.

EXAMPLE **Graph (a) $y = 3 \sin x$ and (b) $y = \cos 3x$.**

Solution:

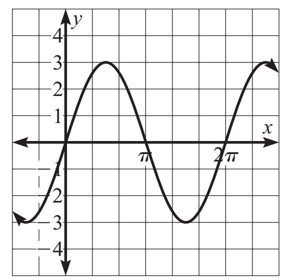
Given nonzero real numbers a and b in functions $y = a \sin bx$ and $y = a \cos bx$, the amplitude of each is $|a|$ and the period of each is $\frac{2\pi}{|b|}$.

a. The amplitude is $a = 3$ and the period is $\frac{2\pi}{b} = \frac{2\pi}{1} = 2\pi$.

Intercepts: $(0, 0); \left(\frac{1}{2} \cdot 2\pi, 0\right) = (\pi, 0); (2\pi, 0)$

Maximum: $\left(\frac{1}{4} \cdot 2\pi, 3\right) = \left(\frac{\pi}{2}, 3\right)$

Minimum: $\left(\frac{3}{4} \cdot 2\pi, -3\right) = \left(\frac{3\pi}{2}, -3\right)$

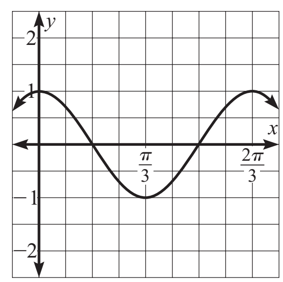


b. The amplitude is $a = 1$ and the period is $\frac{2\pi}{b} = \frac{2\pi}{3}$.

Intercepts: $\left(\frac{1}{4} \cdot \frac{2\pi}{3}, 0\right) = \left(\frac{\pi}{6}, 0\right); \left(\frac{3}{4} \cdot \frac{2\pi}{3}, 0\right) = \left(\frac{\pi}{2}, 0\right)$

Maximums: $(0, 1); \left(\frac{2\pi}{3}, 1\right)$

Minimum: $\left(\frac{1}{2} \cdot \frac{2\pi}{3}, -1\right) = \left(\frac{\pi}{3}, -1\right)$



PRACTICE **Graph the function.**

1. $y = 5 \cos x$ **2.** $y = \sin 2x$ **3.** $y = 3 \sin 4\pi x$

2. Graph a Tangent Function

EXAMPLE **Graph one period of the function $y = 3 \tan 2x$.**

Given nonzero real numbers a and b in the function $y = a \tan bx$, the period is $\frac{\pi}{|b|}$. The vertical asymptotes are odd multiples of $\frac{\pi}{(2|b|)}$. There are no maximum or minimum values, so there is no amplitude.

Solution:

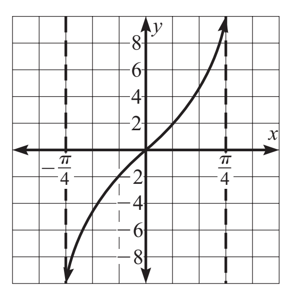
The period is $\frac{\pi}{b} = \frac{\pi}{2}$.

Intercept: $(0, 0)$

Asymptotes: $x = \frac{\pi}{2b} = \frac{\pi}{2 \cdot 2}$ or $\frac{\pi}{4}$;

$$x = -\frac{\pi}{2b} = -\frac{\pi}{2 \cdot 2} \text{ or } -\frac{\pi}{4}$$

Halfway points: $\left(\frac{\pi}{8}, 3\right)$ and $\left(-\frac{\pi}{8}, -3\right)$



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PRACTICE **Graph one period of the function.**

4. $y = 4 \tan x$ **5.** $y = \tan 3x$ **6.** $y = 2 \tan \frac{1}{2}x$

3. Graph Translations and Reflections

Vocabulary

Translation of a trigonometric function A horizontal shift h units, a vertical shift k units, or a combination of both a horizontal shift and a vertical shift in the graph of a function.

Reflection of a trigonometric function A flip across a horizontal line equidistant from the maximum and minimum points on a function's graph.

Midline The horizontal line a function is reflected across.

EXAMPLE **Graph $y = -3 \sin \frac{1}{2}(x + \pi)$.**

The graphs of $y = a \sin b(x - h) + k$ and $y = a \cos b(x - h) + k$ are shifted horizontally h units and vertically k units from their parent functions.

Solution:

Step 1: Identify the amplitude, period, horizontal shift, and vertical shift.

Amplitude: $|a| = |-3| = 3$; Period: $\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi$

Horizontal shift: $h = -\pi$; Vertical shift: $k = 0$

Step 2: Draw the midline of the graph. Since $k = 0$, the midline is the x-axis.

Step 3: Find five key points of $y = |-3| \sin \frac{1}{2}(x + \pi)$.

On the midline $y = k$: $(0 - \pi, 0) = (-\pi, 0)$;

$(2\pi - \pi, 0) = (\pi, 0)$;

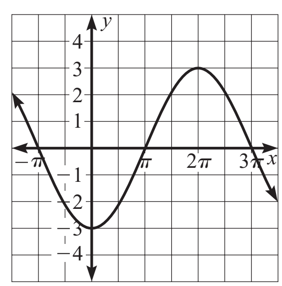
$(4\pi - \pi, 0) = (3\pi, 0)$

Maximum: $(\pi - \pi, 3) = (0, 3)$

Minimum: $(3\pi - \pi, -3) = (2\pi, -3)$

Step 4: Reflect the graph. Since $a < 0$, the graph is reflected in the midline $y = 0$. So, $(0, 3)$ becomes $(0, -3)$ and $(2\pi, -3)$ becomes $(2\pi, 3)$.

Step 5: Draw the graph through the key points.



PRACTICE **Graph the function.**

7. $y = -\cos \frac{1}{2}(x - \pi) + 2$ **8.** $y = 2 \sin\left(x + \frac{\pi}{2}\right) - 3$

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Quiz

Graph one period of the function.

1. $y = 6 \sin x$

2. $y = \cos 2x$

3. $y = \tan \frac{1}{4}x$

4. $y = 3 \cos 4x$

5. $y = 4 \sin \frac{1}{2}x$

6. $y = 2 \tan \pi x$

7. $y = -2 \sin \left(x - \frac{\pi}{2}\right)$

8. $y = 4 \cos \frac{2}{3}x + 1$

9. $y = -\tan \frac{1}{4}(x + \pi) - 2$

10. $y = 3 \sin \frac{1}{3}(x - \pi) + 2$

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BENCHMARK 6
D. Trigonometric Identities

D. Trigonometric Identities (pp. 130–132)

Fundamental trigonometric identities that can be used to simplify expressions, evaluate functions, and help solve equations. Several trigonometric identities are described and applied below.

1. Use Fundamental Trigonometric Identities

Vocabulary

Trigonometric identity An expression or equation involving trigonometric functions that is true for all values of the variable.

EXAMPLE

Trigonometric cofunction identities include:

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$

$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$

$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$

$\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$

Simplify the expression $\dfrac{\cos\left(\frac{\pi}{2} - x\right)}{\tan x}$.

Solution:

$$\frac{\cos\left(\frac{\pi}{2} - x\right)}{\tan x} = \frac{\sin x}{\tan x}$$ **Substitute $\sin x$ for $\cos\left(\frac{\pi}{2} - x\right)$.**

$$= \frac{\sin x}{\frac{\sin x}{\cos x}}$$ **Substitute $\frac{\sin x}{\cos x}$ for $\tan x$.**

$$= \cos x$$ **Simplify.**

PRACTICE

Trigonometric Pythagorean identities include:

$\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Simplify the expression.

1. $\dfrac{\cot x \sec x}{\csc x}$ **2.** $\csc^2 x - \sin x \csc x$ **3.** $\cot\theta(\sec^2\theta - 1)$

2. Use Sum and Difference Formulas

$\sin(a + b) \neq \sin a + \sin b$

$\sin(a - b) \neq \sin a - \sin b$

The same is true for the other trigonometric functions.

Trigonometric functions relating to the **sum and difference of two angles** are as follows:

$\sin(a + b) = \sin a \cos b + \cos a \sin b$ $\sin(a - b) = \sin a \cos b - \cos a \sin b$

$\cos(a + b) = \cos a \cos b - \sin a \sin b$ $\cos(a - b) = \cos a \cos b + \sin a \sin b$

$\tan(a + b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$ $\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$

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EXAMPLE **Find the exact value of $\tan \frac{\pi}{12}$.**

Solution:

$\tan \frac{\pi}{12} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$ — **Substitute $\frac{\pi}{4} - \frac{\pi}{6}$ for $\frac{\pi}{12}$.**

$= \dfrac{\tan \frac{\pi}{4} - \tan \frac{\pi}{6}}{1 + \tan \frac{\pi}{4} \tan \frac{\pi}{6}}$ — **Difference formula for tangent**

$= \dfrac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \cdot \frac{\sqrt{3}}{3}}$ — **Evaluate.**

$= 2 - \sqrt{3}$ — **Simplify.**

EXAMPLE **Find $\sin(a - b)$ given that $\cos a = -\frac{3}{5}$ with $\frac{\pi}{2} < a < \pi$ and $\sin b = \frac{8}{17}$ with $0 < b < \frac{\pi}{2}$.**

Solution:

The sine of an angle in Quadrant II is positive. The cosine of an angle in Quadrant II is negative.

Using a Pythagorean identity and quadrant signs gives $\sin a = \frac{4}{5}$ and $\cos b = \frac{15}{17}$.

$\sin(a - b) = \sin a \cos b - \cos a \sin b$ — **Difference formula for sine**

$= \frac{4}{5} \cdot \frac{15}{17} - \left(-\frac{3}{5}\right) \cdot \frac{8}{17}$ — **Substitute.**

$= \frac{84}{85}$ — **Simplify.**

PRACTICE **Find the exact value of the expression.**

4. $\cos 75°$ **5.** $\tan 105°$ **6.** $\sin \frac{\pi}{12}$

7. Find $\cos(a + b)$ given that $\sin a = \frac{12}{13}$ with $\frac{\pi}{2} < a < \pi$ and $\sin b = -\frac{4}{5}$ with $\frac{3\pi}{2} < b < 2\pi$.

3. Use Double-Angle and Half-Angle Formulas

Double-angle formulas:

$\sin 2a = 2 \sin a \cos a$

$\cos 2a = \cos^2 a - \sin^2 a$

$= 2\cos^2 a - 1$

$= 1 - 2\sin^2 a$

$\tan 2a = \dfrac{2 \tan a}{1 - \tan^2 a}$

Half-angle formulas:

$\sin \frac{a}{2} = \pm\sqrt{\dfrac{1 - \cos a}{2}}$

$\cos \frac{a}{2} = \pm\sqrt{\dfrac{1 + \cos a}{2}}$

$\tan \frac{a}{2} = \dfrac{1 - \cos a}{\sin a}$

$= \dfrac{\sin a}{1 + \cos a}$

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EXAMPLE

Because 15° is in Quadrant I, the value of the sine is positive.

a. Find the exact value of sin 15°.

Solution:

$$\sin 15° = \sin \tfrac{1}{2}(30°) = \sqrt{\frac{1 - \cos 30°}{2}} = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2 - \sqrt{3}}}{2}$$

b. Give $\sin a = \frac{3}{5}$ with $\frac{\pi}{2} < a < \pi$, find $\sin 2a$ and $\sin \frac{a}{2}$.

Solution:

Using a Pythagorean identity and quadrant signs gives $\cos a = -\frac{4}{5}$.

$$\sin 2a = 2 \sin a \cos a = 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) = -\frac{24}{25}$$

$$\sin \frac{a}{2} = \sqrt{\frac{1 - \cos a}{2}} = \sqrt{\frac{1 - \left(-\frac{4}{5}\right)}{2}} = \sqrt{\frac{9}{10}} = \frac{3\sqrt{10}}{10}$$

PRACTICE

Given $\cos a = \frac{3}{5}$ with $\frac{3\pi}{2} < a < 2\pi$, find each value.

8. $\cos 2a$ **9.** $\tan \frac{a}{2}$ **10.** $\cos \frac{a}{2}$

Quiz

Simplify the expression.

1. $\cos \theta \tan \theta$

2. $\cos^2 x + \cos^2 \left(\frac{\pi}{2} - x\right)$

3. $\cos x \left[1 + \cot^2 \left(\frac{\pi}{2} - x\right)\right] - \sec x$

Find the exact value of the expression.

4. $\sin 105°$ **5.** $\cos \frac{11\pi}{12}$ **6.** $\tan 255°$

7. Find $\cos (a - b)$ given that $\sin a = -\frac{4}{5}$ with $\pi < a < \frac{3\pi}{2}$ and $\cos b = -\frac{4}{5}$ with $\frac{\pi}{2} < b < \pi$.

Simplify the expression.

8. $\sin (x - \pi)$ **9.** $\tan (x + \pi)$ **10.** $\cos \left(x + \frac{3\pi}{2}\right)$

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E. Solve Trigonometric Equations (pp. 133–135)

When a trigonometric function is true for all values of the variable, the function is an identity. When a function is true only for some values of the variable, the function is an equation. The methods for finding the true values of the variable in a trigonometric equation are presented below.

1. Solve a Trigonometric Equation

EXAMPLE **Solve $2 \sin x - 1 = 0$.**

Solution:

First isolate sin x on one side of the equation.

$2 \sin x - 1 = 0$ **Write original equation.**

$2 \sin x = 1$ **Add 1 to each side.**

$\sin x = \frac{1}{2}$ **Divide each side by 2.**

One solution of $\sin x = \frac{1}{2}$ over the interval $0 \le x < 2\pi$ is $x = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$. The other solution in this interval is $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$. Since $y = \sin x$ is periodic, there are infinitely many solutions. Using the two solutions above, the general solution is written as follows:

$$x = \frac{\pi}{6} + 2n\pi \text{ or } x = \frac{5\pi}{6} + 2n\pi, \text{ for any integer } n$$

EXAMPLE **Solve $8 \cos^2 x + 1 = 3$ in the interval $0 \le x < 2\pi$.**

Solution:

$8 \cos^2 x + 1 = 3$ **Write original equation.**

$8 \cos^2 x = 2$ **Subtract 1 from each side.**

$\cos^2 x = \frac{1}{4}$ **Divide each side by 8.**

$\cos x = \pm\frac{1}{2}$ **Take square roots of each side.**

It is helpful to memorize the trigonometric values for special angles 0°, 30°, 45°, 60°, 90°, and 180°.

One solution of $\cos x = \pm\frac{1}{2}$ is $x = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$. Another solution is $x = \cos^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{3}$. Over the interval $0 \le x < 2\pi$, the solutions are:

$x = \frac{\pi}{3}$ $\quad$ $x = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$

$x = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$ $\quad$ $x = -\frac{\pi}{3} + 2\pi = \frac{5\pi}{3}$

PRACTICE **Find the general solution of the equation.**

1. $3 \tan x + 4 = 1$ **2.** $4 \sin^2 x + 2 = 5$

Solve the equation in the interval $0 \le x < 2\pi$.

3. $2 \cos x + 2 = 3$ **4.** $3 \tan^2 x - 1 = 8$

BENCHMARK 6 E. Solving Trigonometrics

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2. Solve an Equation with Extraneous Solutions

EXAMPLE **Solve $1 - \sin x = \cos x$ in the interval $0 \le x < 2\pi$.**

Solution:

$1 - \sin x = \cos x$	**Write original equation.**
$(1 - \sin x)^2 = (\cos x)^2$	**Square both sides.**
$1 - 2\sin x + \sin^2 x = \cos^2 x$	**Multiply.**
$1 - 2\sin x + \sin^2 x = 1 - \sin^2 x$	**Pythagorean identity**
$2\sin^2 x - 2\sin x = 0$	**Quadratic form**
$2\sin x(\sin x - 1) = 0$	**Factor out 2 sin *x*.**
$2\sin x = 0$ or $\sin x - 1 = 0$	**Zero product property**
$\sin x = 0$ or $\sin x = 1$	**Solve for sin *x*.**

Over the interval $0 \le x < 2\pi$, $\sin x = 0$ has two solutions: $x = 0$ or $x = \pi$.

Over the interval $0 \le x < 2\pi$, $\sin x = 1$ has one solution: $x = \frac{\pi}{2}$.

Therefore, $1 - \sin x = \cos x$ has three possible solutions: $x = 0, \frac{\pi}{2}$, and π.

It is a good practice to always check solutions in trigonometric equations since it is possible for these equations to contain extraneous solution.

Check Substitute the possible solutions into the original equation and simplify.

$x = 0$:	$1 - \sin 0 = \cos 0$	
	$1 - 0 = 1$	
	$1 = 1$	**Solution checks**
$x = \frac{\pi}{2}$:	$1 - \sin \frac{\pi}{2} = \cos \frac{\pi}{2}$	
	$1 - 1 = 0$	
	$0 = 0$	**Solution checks**
$x = \pi$:	$1 - \sin \pi = \cos \pi$	
	$1 - 0 = -1$	
	$1 \ne -1$	**Solution is extraneous**

The only solutions to $1 - \sin x = \cos x$ in the interval $0 \le x < 2\pi$ are $x = 0$ and $x = \frac{\pi}{2}$.

PRACTICE **Solve the equation in the interval $0 \le x < 2\pi$.**

5. $-\cos^2 x = \sin x \cos^2 x$

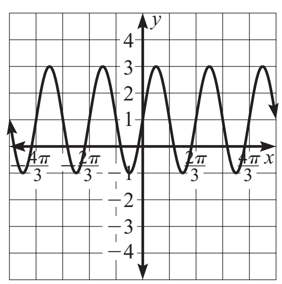
6. $\tan x \sin^2 x - \tan x = 0$

3. Write a Sinusoid

Vocabulary **Sinusoid** A graph of a sine or cosine function.

EXAMPLE **Write a function for the sinusoid.**

In the sinusoid $y = a \sin b(x - h) + k$, $|a|$ is the amplitude, b is used to find the period, h is the horizontal shift, and k is the vertical shift.



Solution:

Step 1: **Find** the maximum value M and the minimum value m. From the graph, $M = 3$ and $m = -1$.

Step 2: **Identify** the vertical shift, k. This value is the mean of M and m. So, $k = 1$.

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Step 3: Decide whether the graph models a sine or cosine function. Since the graph crosses the midline $y = 1$ on the y-axis, the graph is a sine curve with no horizontal shift. So, $h = 0$.

Step 4: Find the amplitude and period. The period $\frac{2\pi}{b} = \frac{2\pi}{3}$ so $b = 3$.

The amplitude $a = \frac{M - m}{2} = \frac{3 - (-1)}{2} = 2$. The graph is not a reflection, so $a = 2$.

The function is $y = 2 \sin 3x + 1$.

PRACTICE **Write a function for the sinusoid.**

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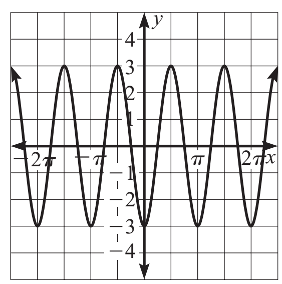

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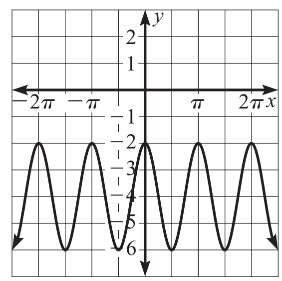


Quiz

Find the general solution of the equation.

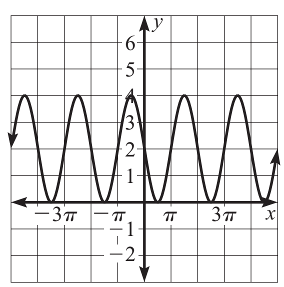
1. $4 \sin x + 4 = 0$ **2.** $8 \cos^2 x + 3 = 7$ **3.** $2 \tan x - 3 = -3$

Solve the equation in the interval $0 \le x < 2\pi$.

4. $6 \sin^2 x + 2 = 5$ **5.** $\cos x \cot x = 2 - \sin x$ **6.** $\sin x + \sqrt{3} \cos x = 1$

Write a function for the sinusoid.

7.



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